The Definite Integral

Introduction

It turns out that the Riemann sums of a continuous function f over an interval $[a, b]$ all approach the same value as n gets large.
In this module, we will define the definite integral of f over $[a, b]$ as the limit of these Riemann sums and introduce the process of integration.

Integration

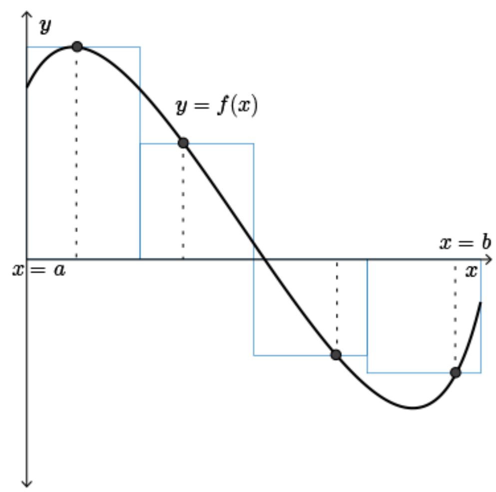
Let $f(x)$ be a continuous function defined on an interval $[a, b]$.
Recall that a Riemann sum of f over $[a, b]$ is a finite sum of the form

$$\sum_{k=1}^{n} f(c_k)\,\Delta x$$

for some n and some choice of sample points $c_1, \ldots, c_n$.

We would like to examine what happens to the Riemann sums as n goes to infinity.

Fix a number n. For the sake of this discussion, let's choose $n = 4$.
Each choice of sample points c_1, c_2, c_3, c_4 corresponds to a (possibly) different Riemann sum of f with 4 subintervals.



Likely, we will have an infinite collection of Riemann sums, all corresponding to $n = 4$.
Recall that each Riemann sum is a **real number**, and every Riemann sum with $n = 4$ is an approximation to the "net area" between f and the x-axis over $[a, b]$ using 4 (signed) rectangles.

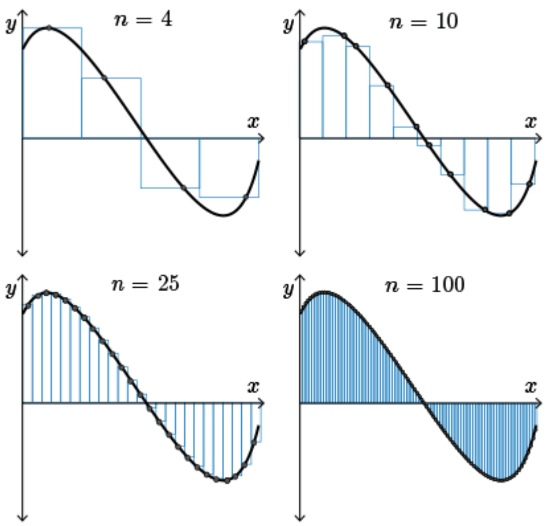
Integration

The same is true for every n. We are interested in what happens to these approximations as n gets larger.

Here, we see examples of Riemann sums for the same function, f, for $n = 4$, $n = 10$, $n = 25$, and $n = 100$.

Once we have reached $n = 100$, the rectangles are so thin that they appear to be more like vertical line segments.

Moreover, since the intervals over which we are choosing our sample points are so small, the actual sample points we select seem less important.



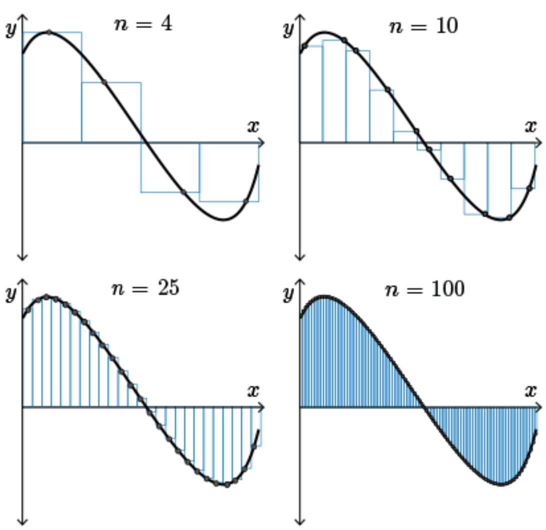
Integration

The same is true for every n. We are interested in what happens to these approximations as n gets larger.

What do we expect as n gets even larger, say $n = 1000$ or $n = 10\,000$?

As n gets very large, we might expect all Riemann sums of the given function, f, to approach some common value, regardless of the particular sample points chosen.

This common limit will represent the "net area" of the region bounded by the curve, f, and the x-axis over the given interval.



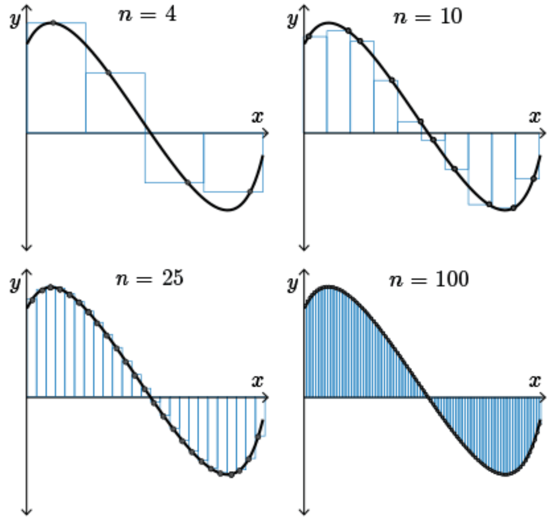
Integration

Question: For which functions, f, do the Riemann sums of f over $[a, b]$ indeed have a common limit, say L?

The limit of the Riemann sums of f is often denoted by

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x$$

If $\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x$ exists, then we say that f is integrable over $[a, b]$.



Definition and Notation

Recall that a Riemann sum of f over $[a, b]$ is a sum of the following form:

$$\sum_{k=1}^{n} f(c_k)\Delta x$$

These sums approximate the net area between the curve $y = f(x)$ and the x-axis over the interval from a to b.
If the collection of Riemann sums approaches a common value as n approaches infinity, then is number is the exact value of the net area.

The **definition integral of f over $[a, b]$** is defined to be the limit of the Riemann sums of f over $[a, b]$, provided that this limit exists. We write

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x$$

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$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x$$

$$\sum \rightsquigarrow \int \qquad\qquad \Delta x \rightsquigarrow dx$$

$$\int_a^b f(x)\ dx$$

- $\int$ is called the **integral sign**.
- a is the **lower limit** of integration.
- b is the **upper limit** of integration.
- $f(x)$, the function, is called the **integrand**.
- dx tells us that we are integrating with respect to x, the **variable of integration**.
- The integral, $\int_a^b f(x)\ dx$, is read "the **integral** of f from a to b".
- When you find the value of the integral, you have **evaluated** the integral.

Integration

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

This means that if f is continuous, then the Reimann sums for f all have a common limit.
Take a second to go back to our earlier discussion, and convince yourself that a continous function that has a nice well-defined area between the function and the x-axis, will be integrable.

Aside

From our discussion, you may be tempted to think that an integrable function must be continuous, but this is not true.
We can define Riemann sums for discontinuous functions and there are simple examples of functions that are *not* continuous but *are* integrable.
An example is the following piecewise function:

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

When picturing what an integrable function "looks like" in general, it is best not to picture a continuous function, but a function that bounds a "well-defined net area".
These are not exactly the same thing.
Can you see how the piecewise function given is not continuous over $[-1, 1]$ but still defines a precise net area over the interval?

With this in mind, can you think of a function that is NOT integrable?
This may be tough as most of the functions we are familiar with are, in fact, integrable functions!

Investigation

Investigate the relationship between Riemann sums and net areas under curves.
Think about what happens to the Riemann sums, graphically and numerically, as n approaches infinity.
How large does n have to be for the Riemann sum to approximate the definite integral to a particular accuracy?
Experiment with sample points being chosen as left endpoints, right endpoints, midpoints, and random sample points.

Examples

Example 1

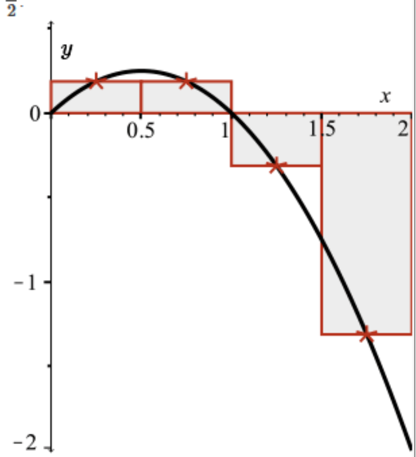
a. Approximate the definite integral $\int_0^2 (x - x^2)\, dx$ by using a Riemann sum with $n = 4$ and taking the sample values (c_k's) to be the midpoints of the subintervals.

Solution

We have $n = 4$ and so the subinterval length is $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$.
The four subintervals are $\left[0, \frac{1}{2}\right]$, $\left[\frac{1}{2}, 1\right]$, $\left[1, \frac{3}{2}\right]$, and $\left[\frac{3}{2}, 2\right]$. The four midpoints are $c_1 = \frac{1}{4}$, $c_2 = \frac{3}{4}$, $c_3 = \frac{5}{4}$, and $c_4 = \frac{7}{4}$.
Therefore, our Riemann sum is

$$\begin{aligned} M_4 &= \sum_{k=1}^{4} f(c_k)\,\Delta x \\ &= f\left(\frac{1}{4}\right)\Delta x + f\left(\frac{3}{4}\right)\Delta x + f\left(\frac{5}{4}\right)\Delta x + f\left(\frac{7}{4}\right)\Delta x \\ &= \left[\frac{3}{16} + \frac{3}{16} + \left(-\frac{5}{16}\right) + \left(-\frac{21}{16}\right)\right]\Delta x \\ &= \left[-\frac{20}{16}\right]\frac{1}{2} = -\frac{10}{16} = -0.675 \end{aligned}$$

We conclude that $\int_0^2 (x - x^2)\, dx \approx -0.675$.



Examples

Example 1

b. Evaluate $\int_0^2 (x - x^2)\, dx$.

Solution

Recall that the definition of the integral is the limit of the Reimann sums.
First, notice that this function is continuous on $[0, 2]$ and so it is integrable over $[0, 2]$.
This means that all of the Riemann sums of the function $f(x) = x - x^2$ over $[0, 2]$ have the same limit, so we can choose to work with whichever sample points we would like.
Let's keep things simple and use the right endpoints of the intervals. Let's find a formula for R_n.

Since $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$, we have:

$$x_0 = 0, \quad x_1 = x_0 + \frac{2}{n} = 0 + \frac{2}{n} = \frac{2}{n}, \quad x_2 = x_1 + \frac{2}{n} = \frac{2}{n} + \frac{2}{n} = \frac{4}{n}, \ldots$$

In general, $x_k = \dfrac{2k}{n}$ for $k = 0, 1, \ldots, n$.
We are using the right endpoints as our sample points, so the sample points are $c_1 = x_1$, $c_2 = x_2, \ldots, c_n = x_n$.
In general, we have $c_k = x_k$ for $k = 1, 2, \ldots, n$.

Examples

Example 1

b. Evaluate $\int_0^2 (x - x^2)\, dx$.

Solution

Therefore, our Riemann sum is given by the following equations and their corresponding sigma notations:

$$R_n = f(c_1)\,\Delta x + f(c_2)\,\Delta x + \cdots + f(c_n)\,\Delta x \qquad \sum_{k=1}^{n} f(c_k)\,\Delta x$$

$$= f(x_1)\,\Delta x(x_2)\,\Delta x + \cdots f(x_n)\,\Delta x \qquad = \sum_{k=1}^{n} f(x_k)\,\Delta x$$

$$= \left(f\left(\frac{2}{n}\right)\right)\left(\frac{2}{n}\right) + \left(f\left(\frac{4}{n}\right)\right)\left(\frac{2}{n}\right) + \cdots + \left(f\left(\frac{2n}{n}\right)\right)\left(\frac{2}{n}\right) \qquad = \sum_{k=1}^{n}\left(f\left(\frac{2k}{n}\right)\right)\left(\frac{2}{n}\right)$$

$$= \frac{2}{n}\left[f\left(\frac{2}{n}\right) + f\left(\frac{4}{n}\right) + \cdots + f\left(\frac{2n}{n}\right)\right] \qquad = \frac{2}{n}\left[\sum_{k=1}^{n} f\left(\frac{2k}{n}\right)\right]$$

$$= \frac{2}{n}\left[\left(\frac{2}{n} - \left(\frac{2}{n}\right)^2\right) + \left(\frac{4}{n} - \left(\frac{4}{n}\right)^2\right) + \cdots + \left(\frac{2n}{n} - \left(\frac{2n}{n}\right)^2\right)\right] \qquad = \frac{2}{n}\left[\sum_{k=1}^{n}\left(\frac{2k}{n} - \left(\frac{2k}{n}\right)^2\right)\right]$$

Examples

Example 1

b. Evaluate $\int_0^2 (x - x^2)\,dx$.

Solution

$$= \frac{2}{n}\left[\left(\frac{2}{n} - \left(\frac{2}{n}\right)^2\right) + \left(\frac{4}{n} - \left(\frac{4}{n}\right)^2\right) + \cdots + \left(\frac{2n}{n} - \left(\frac{2n}{n}\right)^2\right)\right] = \frac{2}{n}\left[\sum_{k=1}^{n}\left(\frac{2k}{n} - \left(\frac{2k}{n}\right)^2\right)\right]$$

$$= \frac{2}{n}\left[\frac{2+4+\cdots+2n}{n} - \frac{2^2+4^2+\cdots+(2n)^2}{n^2}\right] \qquad = \frac{2}{n}\left[\frac{1}{n}\sum_{k=1}^{n}2k - \frac{1}{n^2}\sum_{k=1}^{n}(2k)^2\right]$$

$$= \frac{2}{n}\left[\frac{2}{n}(1+2+\cdots+n) - \frac{4}{n^2}(1^2+2^2+\cdots+n^2)\right] \qquad = \frac{2}{n}\left[\frac{2}{n}\sum_{k=1}^{n}k - \frac{4}{n^2}\sum_{k=1}^{n}k^2\right]$$

$$= \frac{4}{n^2}(1+2+\ldots+n) - \frac{8}{n^3}(1^2+2^2+\ldots+n^2) \qquad = \frac{4}{n^2}\sum_{k=1}^{n}k - \frac{8}{n^3}\sum_{k=1}^{n}k^2$$

$$= \frac{4}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \frac{2(n+1)}{n} - \frac{4}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right)$$

Examples

Example 1

b. Evaluate $\int_0^2 (x - x^2)\,dx$.

Solution

$$R_n = \frac{2(n+1)}{n} - \frac{4}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right)$$

Therefore,

$$\begin{aligned}\int_0^2 (x - x^2)\,dx &= \lim_{n\to\infty} R_n \\ &= \lim_{n\to\infty}\left[\frac{2(n+1)}{n} - \frac{4}{3}\left(\frac{2n^2+3n+1}{n^2}\right)\right] \\ &= 2 - \frac{4}{3}(2) \\ &= -\frac{2}{3} \approx -0.667\end{aligned}$$

It turns out that our estimate of -0.675, using only 4 rectangles, wasn't too far off!

Challenge Example

Given that $\lim_{n\to\infty} \frac{1/n}{e^{1/n}-1} = 1$, evaluate the integral $\int_1^2 e^x\,dx$ by finding $\lim_{n\to\infty} L_n$.

Solution

Here, we have no choice but to find the limit of the Riemann sums.

As e^x is continuous, it is integrable and so we can really choose any sample points we would like.
Here, we are asked to use the Riemann sums of the form L_n, using the left endpoints as the sample points.

We have $x_0 = 1$, $x_n = 2$, and $\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$, so

$$x_0 = 1, \quad x_1 = 1 + \frac{1}{n}, \quad x_2 = 1 + \frac{2}{n}, \quad \ldots \quad x_{k-1} = 1 + \frac{k-1}{n}, \quad \ldots \quad x_n = 1 + \frac{n}{n} = 2$$

$$\begin{aligned}
L_n &= \sum_{k=1}^{n} f(c_k)\,\Delta x = \sum_{k=1}^{n} f(x_{k-1})\,\Delta x \\
&= \sum_{k=1}^{n} \left(e^{1+(k-1)/n}\right)\left(\frac{1}{n}\right) \\
&= \frac{1}{n}\left(e^{1+0/n}\right) + \frac{1}{n}\left(e^{1+1/n}\right) + \frac{1}{n}\left(e^{1+2/n}\right) + \cdots + \frac{1}{n}\left(e^{1+(n-1)/n}\right) \\
&= \frac{1}{n}\left[e^1 e^{0/n} + e^1 e^{1/n} + e^1 e^{2/n} + \cdots + e^1 e^{(n-1)/n}\right] \qquad \text{since } e^{a+b} = e^a e^b \\
&= \frac{e}{n}\left[1 + e^{1/n} + e^{2/n} + \cdots + e^{(n-1)/n}\right]
\end{aligned}$$

You may recognize the sum in the brackets as a geometric sum!
The first term is $a = 1$ and the common ratio is $r = e^{1/n}$.
If we plug these values into the formula for a geometric sum

$$a + ar + ar^2 + \ldots + ar^{n-1} = a\left(\frac{1-r^n}{1-r}\right)$$

we get

$$1 + e^{1/n} + e^{2/n} + \cdots + e^{n-1/n} = 1\left(\frac{1-\left(e^{1/n}\right)^n}{1-e^{1/n}}\right) = \frac{1-e}{1-e^{1/n}}$$

Therefore, we have

$$L_n = \frac{e}{n}\left(\frac{1-e}{1-e^{1/n}}\right) = (e-e^2)\,\frac{1/n}{1-e^{1/n}} = (e^2-e)\,\frac{1/n}{e^{1/n}-1}$$

and

$$\int_1^2 e^x\,dx = \lim_{n\to\infty} L_n = \lim_{n\to\infty}(e^2-e)\,\frac{1/n}{e^{1/n}-1} = (e^2-e)\lim_{n\to\infty}\frac{1/n}{e^{1/n}-1} = (e^2-e)(1) = e^2-e$$

Believe it or not, we will soon learn a way to evaluate this integral in a single line!