Reflections

Introduction

Previously, we noted that a transformation changes one or more of the location, shape, size, or orientation of an object in the coordinate plane.

In particular, a translation changed the location of an object or function in the coordinate plane but left all other attributes the same.

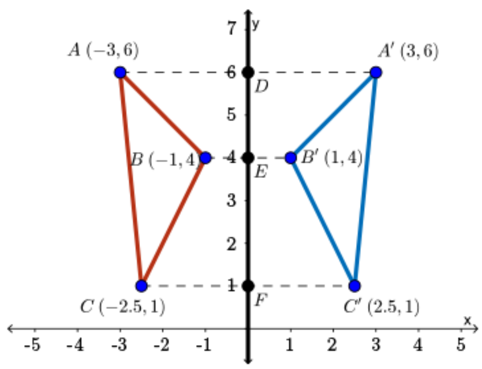
In this module we will examine specific types of reflections:

- reflections in the x-axis,
- reflections in the y-axis, and
- reflections in both the x-axis and the y-axis.

Properties of Reflections

On the graph, $\triangle ABC$ is reflected in the y-axis. The image $\triangle A'B'C'$ is shown.

We will use this specific diagram to make some general observations about reflections.



The size and shape of an object or function is not altered by a reflection.

The location and orientation are changed.

On the pre-image, from A to B to C, we move in a clockwise direction. On the image, from A' to B' to C', we move in a counter-clockwise direction.

The line of reflection is a right bisector of a line segment connecting any pre-image point to its corresponding image point.

In our example, the line of reflection is the y-axis.

We note that it is is perpendicular to AA', BB', and CC'.

Also, AA', BB', and CC' intersect the y-axis at D, E, and F, respectively, such that $AD = DA'$, $BE = EB'$ and $CF = FC'$.

Any point on the line of reflection is unaffected by the reflection. These points are said to be **invariant**. In our example, there are no invariant points.

Properties of Reflections

Consider some function $y = f(x)$.

- If $g(x) = -f(x)$, how are the graphs of $f(x)$ and $g(x)$ related?
- If $h(x) = f(-x)$, how are the graphs of $f(x)$ and $h(x)$ related?
- If $k(x) = -f(-x)$, how are the graphs of $f(x)$ and $k(x)$ related?

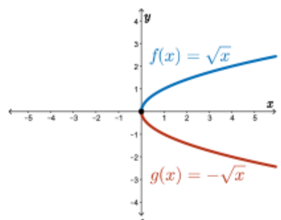
To investigate these questions, use the following worksheet.

When you have completed the investigation, you should be able to answer the above questions.

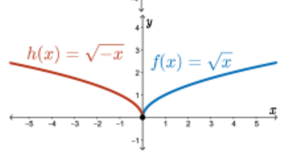
Vertical and Horizontal Reflections

You should have discovered the following after doing the investigation:

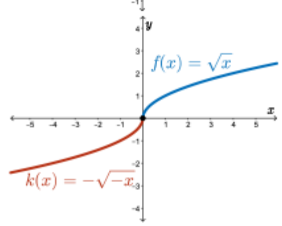
- If $g(x) = -f(x)$, then $g(x)$ is a reflection of $f(x)$ in the x-axis. If $f(x) = \sqrt{x}$ and $g(x) = -f(x)$, then $g(x) = -\sqrt{x}$ and is a reflection of $f(x)$ in the x-axis. Since $(0,0)$ is on the x-axis, $(0,0)$ is an invariant point.



- If $h(x) = f(-x)$, then $h(x)$ is a reflection of $f(x)$ in the y-axis. If $f(x) = \sqrt{x}$ and $h(x) = f(-x)$, then $h(x) = \sqrt{-x}$ and is a reflection of $f(x)$ in the y-axis. Since $(0,0)$ is on the y-axis, $(0,0)$ is an invariant point.



- If $k(x) = -f(-x)$, then $k(x)$ is a reflection of $f(x)$ in the x-axis and the y-axis. If $f(x) = \sqrt{x}$ and $k(x) = -f(-x)$, then $k(x) = -\sqrt{-x}$ and is a reflection of $f(x)$ in the x-axis and the y-axis. The point $(0,0)$ remains invariant through both reflections.



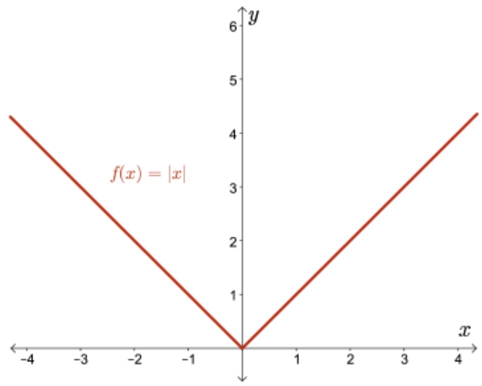
Even Functions

When a function is reflected in the y-axis and the y-axis is also its axis of symmetry, the reflection will leave the original function unchanged.

For example, if $f(x) = x^2$ is reflected in the y-axis (its axis of symmetry), then the resulting image is $g(x) = f(-x) = (-x)^2 = x^2 = f(x)$. Note that $f(x) = f(-x)$.

In general, when $f(x) = f(-x)$ for all values of x in the domain, the function is said to be an **even function**. More formally, an even function is symmetrical about the y-axis and will map onto itself when reflected in the y-axis.

Since $f(x) = |x|$ has the y-axis as its axis of symmetry, then a reflection in the y-axis will produce an image identical to the pre-image, so $f(x) = |x|$ is also an even function.



Odd Functions

When $f(x) = \frac{1}{x}$ is reflected in either the x-axis or the y-axis, an interesting thing happens.

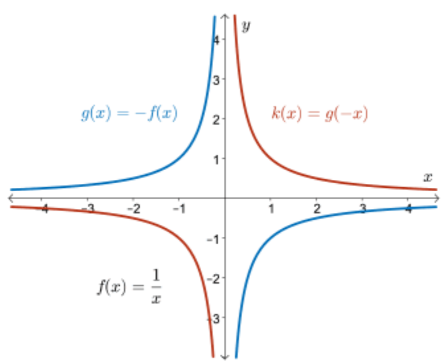
Let $g(x)$ be the image of $f(x) = \frac{1}{x}$ after being reflected in the x-axis. Then $g(x) = -f(x) = -\frac{1}{x}$.

Let $h(x)$ be the image of $f(x) = \frac{1}{x}$ after being reflected in the y-axis. Then
$h(x) = f(-x) = \frac{1}{(-x)} = -\frac{1}{x} = g(x)$.

In general, when $f(-x) = -f(x)$ for all values of x in the domain, the function is called an **odd function**.

An odd function possesses rotational symmetry about the origin. That is, rotating an odd function 180^o about the origin maps the function onto itself.

The same effect is achieved by first reflecting the graph in the x-axis and then reflecting the image in the y-axis, or vice versa.

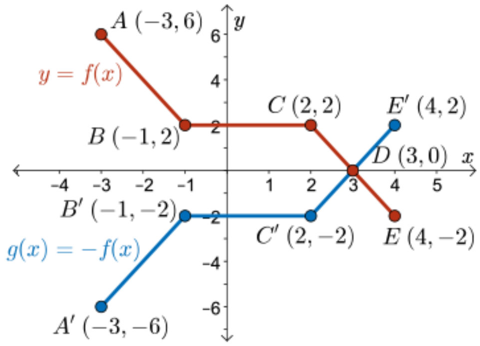


Mapping Notation

Example 1

The graph of $g(x)$ has been obtained by reflecting $f(x)$ in the x-axis. Determine a mapping to describe this reflection.

Solution



Map the points on $f(x)$ to their images on $g(x)$.
Look for the mapping that is applied to each pre-image point to obtain each corresponding image point.

$$A\ (-3,6) \to A'\ (-3,-6)$$
$$B\ (-1,2) \to B'\ (-1,-2)$$
$$C\ (2,2) \to C'\ (2,-2)$$
$$D\ (3,0) \to D'\ (3,0)$$
$$E\ (4,-2) \to E'\ (4,2)$$

Notice that the x-coordinates of both the pre-image and image points are the same.
The y-coordinate of each image point is the opposite of its corresponding pre-image point.
The required mapping is $(x, y) \to (x, -y)$.
Notice that the point $D\ (3, 0)$ is on the x-axis and is invariant.

Mapping Notation

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The graph of $g(x)$ has been obtained by reflecting $f(x)$ in the x-axis. Determine a mapping to describe this reflection.

Note

In a very similar way, we could show that a reflection in the y-axis can be represented using the mapping $(x, y) \to (-x, y)$.
A reflection in both the x-axis and the y-axis can be represented using the mapping $(x, y) \to (-x, -y)$.

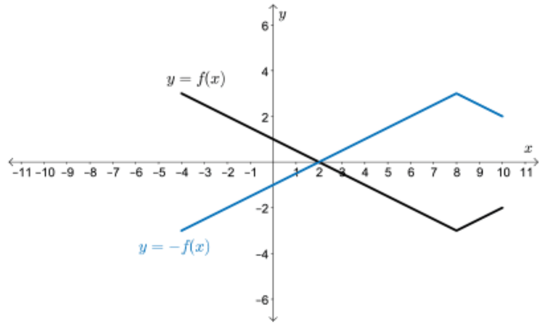
Sketch the Function

Example 2

Given the function $y = f(x)$ shown on the grid, sketch $y = -f(x), y = f(-x)$, and $y = -f(-x)$.

Solution

To graph $y = -f(x)$, note that the transformation is a reflection of $y = f(x)$ in the x-axis.
We could list key points from the pre-image. Then, using the mapping $(x, y) \to (x, -y)$, we could transform the points and plot them to create the image.
Or we could work directly from the graph, reflecting key points on the pre-image in the x-axis to create the image.
Notice that the point $(2, 0)$ is invariant since it is on the x-axis.



Sketch the Function

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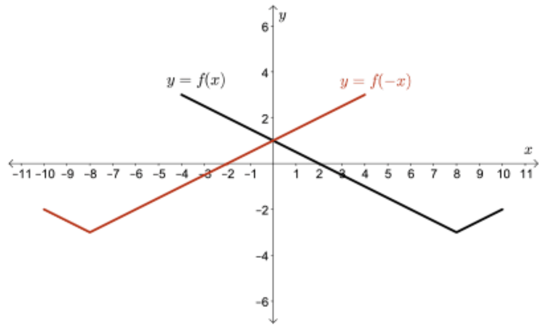
Solution

To graph $y = f(-x)$, note that the transformation is a reflection of $y = f(x)$ in the y-axis.
Going directly to the graph and reflecting key points on the pre-image in the y-axis, we are able to create the image.
The point $(0, 1)$ is invariant since it is on the y-axis.



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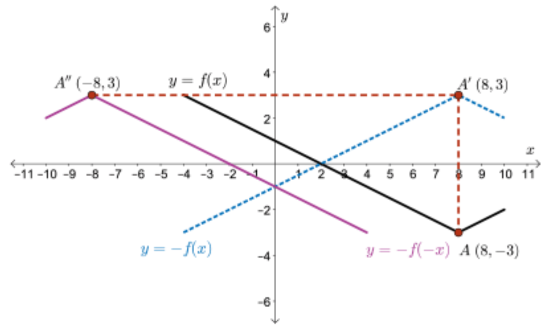


Solution

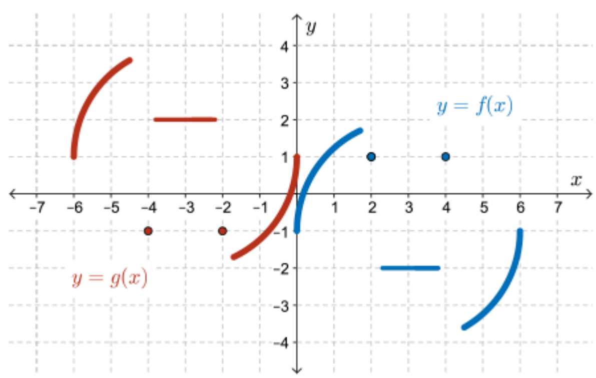
To graph $y = -f(-x)$, recognize that the transformation is a reflection of $y = f(x)$ in both the x-axis and the y-axis.
We can perform the reflections in either order (You may wish to verify this for yourself!).
We can show a reflection in the x-axis and then reflect the image in the y-axis or we can go directly from $y = f(x)$ to $y = -f(-x)$ by transforming key points one at a time.

For example, the point $A\,(8, -3)$ reflects to $A'\,(8, 3)$ after reflecting in the x-axis.
Then $A'\,(8, 3)$ reflects to $A''\,(-8, 3)$ after reflecting in the y-axis. Other key points on the pre-image $y = f(x)$ could be transformed in a similar manner.

Determine the Equation

Example 3

The graph shows two functions: the pre-image $f(x)$ and image $g(x)$. Determine an equation for $g(x)$ in terms of $f(x)$.



Solution

The "eyes" on $f(x)$ are located at about $(2, 1)$ and $(4, 1)$.
The corresponding image points are $(-2, -1)$ and $(-4, -1)$.
It would appear that $g(x)$ is a reflection of $f(x)$ in both the x-axis and the y-axis.
You would discover that, by mapping key points from the pre-image to the image, each of the x-coordinates of the selected pre-image points is the opposite of the x-coordinate of the corresponding image point.
The same is true for the y-coordinates.
This would confirm that $g(x)$ is a reflection of $f(x)$ in both the x-axis and the y-axis.
Therefore, it follows that $g(x) = -f(-x)$.

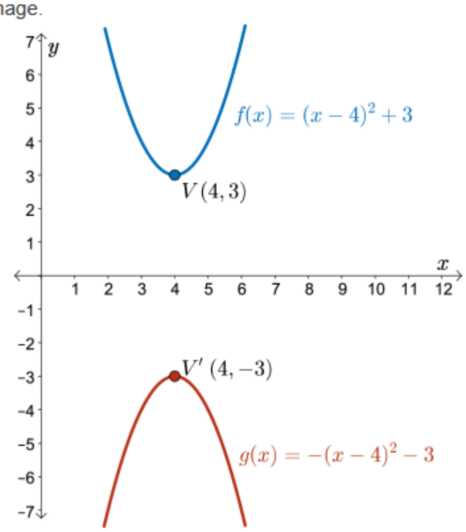
Determine the Equation

Example 4

The function $f(x) = (x-4)^2 + 3$ is reflected in the x-axis. Determine the equation of the image. Sketch both the pre-image and the image. State the domain and range of the image.

Solution

Let $g(x)$ be the image of $f(x)$ after being reflected in the x-axis.
$f(x)$ is a parabola that opens up, is congruent to $y = x^2$, and has vertex at $(4, 3)$.
When reflected in the x-axis, the image would open down, be congruent to $y = x^2$, and have vertex $(4, -3)$.
From our prior knowledge of parabolas, the image equation would be $g(x) = -(x-4)^2 - 3$.
We will verify this using the fact that $g(x) = -f(x)$, since the image is a reflection in the x-axis.
Then, $g(x) = -f(x) = -\left[(x-4)^2 + 3\right] = -(x-4)^2 - 3$.
This confirms our "prediction." The domain of the image is $\{x \mid x \in \mathbb{R}\}$. The range of the image is $\{y \mid y \leq -3, y \in \mathbb{R}\}$.



Determine the Equation

Example 5

The function $f(x) = \sqrt{x+3} - 4$ is reflected in the line $x = 1$. Determine the equation of the image.

Solution

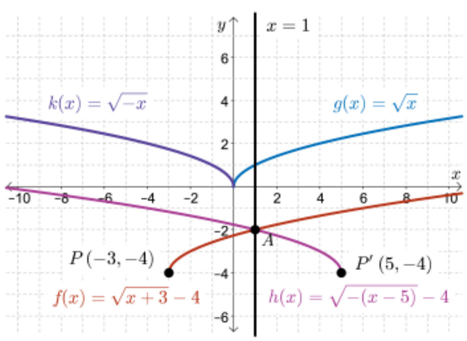
$f(x)$ is a translation of the function $g(x) = \sqrt{x}$ horizontally 3 units left and vertically 4 units down.
We can sketch $g(x)$, $f(x)$ and the vertical line $x = 1$.

The point $(1, -2)$ is on both $f(x)$ and the reflection line $x = 1$. Therefore, it will be invariant.

The point $P\,(-3, -4)$ on $f(x)$ will be reflected to $P'\,(5, -4)$ since $x = 1$ is the right bisector of PP'.

Sketch the image curve. It will be congruent to $y = \sqrt{-x}$. Since the image is congruent to $y = \sqrt{-x}$, we can apply a translation so that $(0, 0)$ is mapped to $(5, -4)$.

The equation of the image will be $h(x) = \sqrt{-(x-5)} - 4$.



Summary

Throughout this lesson, we discussed

- reflections in the x-axis,
- reflections in the y-axis, and
- reflections in both the x-axis and the y-axis.