Solving Polynomial Equations

Solving Quadratic Equations

Quadratics are 2nd degree polynomials.
One way to determine the roots of a quadratic equation (or the zeros of a quadratic function) is to factor the quadratic.

For example, to find the zeros (or x-intercepts) of the given parabola, we would set $y = 0$ and factor.

$$y = 2x^2 + 3x - 20$$
$$0 = (2x - 5)(x + 4)$$
$$2x - 5 = 0 \quad \text{or} \quad x + 4 = 0$$

Therefore, $x = \frac{5}{2}$ and $x = -4$ are the **roots**, or **solutions**, to the equation $0 = 2x^2 + 3x - 20$,
and $x = \frac{5}{2}$ and $x = -4$ are the **zeros**, or **x-intercepts**, of the function $y = 2x^2 + 3x - 20$.

Solving Higher Degree Equations

Example 1

Solve $2x^3 + 8x^2 - 3x - 12 = 0$.

Solution

The first step in solving polynomial equations of degree 2 or higher is to ensure that all terms are to one side of the equation, leaving 0 on the other side.
Once factored, we can then rely on the premise that the product of factors is equal to zero when one of the factors is zero.

$$2x^3 + 8x^2 - 3x - 12 = 0$$

Since $(2)(-12) = (8)(-3)$, this cubic can be factored by grouping.

$$2x^3 + 8x^2 - 3x - 12 = 0$$
$$2x^2(x + 4) - 3(x + 4) = 0$$
$$(x + 4)(2x^2 - 3) = 0$$

Thus, $x + 4 = 0$ or $2x^2 - 3 = 0$.

Solving Higher Degree Equations

Example 1

Solve $2x^3 + 8x^2 - 3x - 12 = 0$.

Solution

$$\begin{aligned} x + 4 &= 0 \\ x &= -4 \end{aligned}$$

$$\begin{aligned} 2x^2 - 3 &= 0 \\ 2x^2 &= 3 \\ x^2 &= \frac{3}{2} \\ x &= -\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}} \end{aligned}$$

Although the radical answer is correct, good mathematical practice is to remove the radical values from the denominator. This is called rationalizing the denominator.

We rationalize the denominator by multiplying the numerator and denominator by $\sqrt{2}$.

$$\begin{aligned} x &= -\frac{\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ x &= -\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2} \end{aligned}$$

Therefore, the roots are $x = -4$ and $\pm\frac{\sqrt{6}}{2}$.

Solving Higher Degree Equations

Example 2

Solve $2x^2(x - 5) = 12x - 48$.

Solution

$$2x^3 - 10x^2 = 12x - 48$$

Expand and collect all terms to one side of the equation.

$$2x^3 - 10x^2 - 12x + 48 = 0$$

$$2(x^3 - 5x^2 - 6x + 24) = 0$$

Remove any common factors.

Let $P(x) = x^3 - 5x^2 - 6x + 24$.

Use the rational root theorem and the factor theorem on the remaining factor, $P(x)$.

Test values for the factor theorem include the following: $\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$.

$P(2) = 0$. Therefore, $(x - 2)$ is a factor.

Solving Higher Degree Equations

Example 2

Solve $2x^2(x-5) = 12x - 48$.

Solution

$2(x^3 - 5x^2 - 6x + 24) = 0$

$2(x-2)(\blacksquare x^2 + \blacksquare x + \blacksquare) = 0$

The corresponding quadratic factor can be found using long division, synthetic division, or the "have and need" method.

$2(x-2)(x^2 + \blacksquare x + \blacksquare) = 0$

Using the "have and need" method, the first term of the quadratic factor must be x^2 since $x(x^2) = x^3$.

$2(x-2)(x^2 + \blacksquare x - 12) = 0$

The last term must be -12 since $-2(-12) = 24$.

Expand what we have so far.

Have: $x^3 - 2x^2 - 12x + 24$

Need: $-3x^2 + 6x = -3x(x-2)$

$2(x-2)(x^2 - 3x - 12) = 0$

The middle term must be $-3x$.

$x - 2 = 0$ or $x^2 - 3x - 12 = 0$

Solving Higher Degree Equations

Example 2

Solve $2x^2(x-5) = 12x - 48$.

Solution

$$x - 2 = 0 \quad \text{or} \quad x^2 - 3x - 12 = 0$$

$$x = 2 \quad \text{or} \quad x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-12)}}{2(1)}$$

$$x = 2 \quad \text{or} \quad x = \frac{3 \pm \sqrt{57}}{2}$$

Therefore, the 3 roots to the equation are $x = 2$, $x \approx 5.27$, and $x \approx -2.27$.

If we consider the related cubic function of this polynomial,

$$y = 2x^3 - 10x^2 - 12x + 48$$

the x-intercepts of the function would be obtained by setting $y = 0$ and solving the equation.
Hence, the zeros of the function would also be $x = 2$, $x \approx 5.27$, and $x \approx -2.27$.

Solving Higher Degree Equations

Example 2

Solve $2x^2(x-5) = 12x - 48$.

Solution

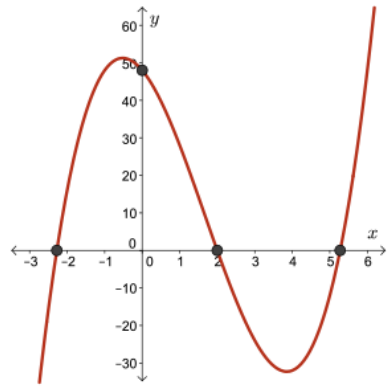
$y = 2x^3 - 10x^2 - 12x + 48$
Zeros: $x = 2$, $x \approx 5.27$, and $x \approx -2.27$
The leading coefficient of the cubic is positive, so

$$x \to \infty, y \to \infty$$

and

$$x \to -\infty, y \to -\infty$$

The y-intercept is 48 (the constant term of the polynomial), which is found by setting $x = 0$.
The three zeros are of multiplicity 1. Therefore, the curve will pass directly through the axis at each zero.
The exact location of the turning points of the curve can be determined using calculus techniques, but this is beyond the scope of this course.



Solving Higher Degree Equations

Example 3

Sketch a possible graph of $f(x) = -x^3 - 4x + 5$ using the zeros, y-intercept, and end behaviour of the function.

Solution

First, we find the zeros. Set $f(x) = 0$, and then we determine the roots of the cubic equation.

$$0 = -x^3 - 4x + 5$$

It may be easy to factor this cubic by first factoring the leading coefficient, -1, from each term.

$$0 = -(x^3 + 4x - 5)$$

$f(1) = 0$. Therefore, $(x - 1)$ is a factor of $f(x)$.

We can then use long division, synthetic division, or the "have and need" method to find the corresponding factor.

$$0 = -(x - 1)(x^2 + x + 5)$$

Solving Higher Degree Equations

Example 3

Sketch a possible graph of $f(x) = -x^3 - 4x + 5$ using the zeros, y-intercept, and end behaviour of the function.

Solution

$$0 = -(x-1)(x^2 + x + 5)$$

The quadratic factor, $x^2 + x + 5$, is non-factorizable, so $x - 1 = 0$ or $x^2 + x + 5 = 0$.

$$x = 1 \quad \text{or} \quad x = \frac{-1 \pm \sqrt{1^2 - 4(1)(5)}}{2(1)}$$
$$= \frac{-1 \pm \sqrt{-19}}{2}$$

These non-real roots can be expressed as

$$\frac{-1 \pm \sqrt{-19}}{2} = \frac{-1 \pm \sqrt{19i^2}}{2} = \frac{-1 \pm i\sqrt{19}}{2}$$

using i, where $i^2 = -1$.

Therefore, the roots are $x = 1$ and $x = \frac{-1 \pm i\sqrt{19}}{2}$.

The values containing i are in the complex number system: a number system containing both real numbers and imaginary numbers. Imaginary numbers are formed from the square root of negative values.

Non-real values for zeros do not result in x-intercepts for the function sketched in the (x, y)-plane.

Solving Higher Degree Equations

Example 3

Sketch a possible graph of $f(x) = -x^3 - 4x + 5$ using the zeros, y-intercept, and end behaviour of the function.

Solution

Graphically, this means that there is only one x-intercept at $(1, 0)$. The y-intercept is $(0, 5)$.

This is a cubic function with negative leading coefficient, -1, hence

$$x \to \infty, y \to -\infty$$

and

$$x \to -\infty, y \to \infty$$

With only this information, we are unsure of whether the cubic has two turning points or a point of inflection (no turning point).

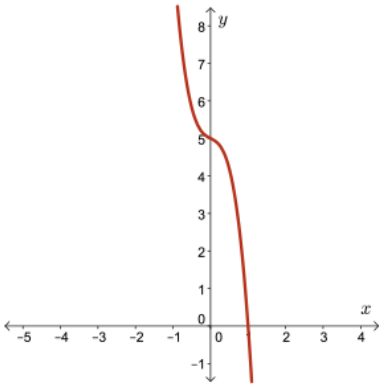
Solving Higher Degree Equations

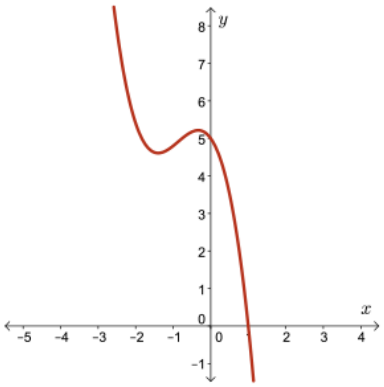
Example 3

Sketch a possible graph of $f(x) = -x^3 - 4x + 5$ using the zeros, y-intercept, and end behaviour of the function.

Solution

The following two graphs satisfy the information we have for the function, but they may not accurately depict the graph of this specific function. More information is needed.





Solving Higher Degree Equations

Example 4

Given that the equation $x^3 + kx^2 - 3x + 18 = 0$ has 2 equal real roots, determine the value of k and all roots of this equation.

Solution

Fact: A cubic equation has 3 roots.
Fact: Non-real (complex) roots occur in conjugate pairs (i.e., $a \pm bi$).
Let the 3 roots of the cubic equation be $x = p$, $x = p$, and $x = r$, where p and r are real.
Then

$$\begin{aligned}(x-p)(x-p)(x-r) &= x^3 + kx^2 - 3x + 18\\ (x^2 - 2px + p^2)(x-r) &= x^3 + kx^2 - 3x + 18\\ x^3 - rx^2 - 2px^2 + 2prx + p^2x - p^2r &= x^3 + kx^2 - 3x + 18\\ x^3 - (2p+r)x^2 + (2pr + p^2)x - p^2r &= x^3 + kx^2 - 3x + 18\end{aligned}$$

We equate the coefficients.

$$-(2p + r) = k \qquad (1)$$
$$2pr + p^2 = -3 \qquad (2)$$
$$-p^2r = 18 \qquad (3)$$

Since $-p^2r = 18$, $p \neq 0$ and $r \neq 0$. Rearranging (3), we have $r = -\dfrac{18}{p^2}$.

Solving Higher Degree Equations

Example 4

Given that the equation $x^3 + kx^2 - 3x + 18 = 0$ has 2 equal real roots, determine the value of k and all roots of this equation.

Solution

Substituting (3), $r = -\frac{18}{p^2}$, into (2), $2pr + p^2 = -3$, gives

$$\begin{aligned} 2p\left(-\frac{18}{p^2}\right) + p^2 &= -3 \\ -\frac{36}{p} + p^2 &= -3 \\ \text{Multiplying both sides by } p,\quad -36 + p^3 &= -3p \\ p^3 + 3p - 36 &= 0 \end{aligned}$$

Using the factor theorem, the polynomial, $p^3 + 3p - 36$, equals zero when $p = 3$. Therefore, $p - 3$ is a factor. To find the corresponding quadratic factor, we can use either long division, synthetic division, or the "have and need" method.

$$(p-3)(p^2 + 3p + 12) = 0$$

Therefore, $p = 3$ or $p = \frac{-3 \pm \sqrt{-39}}{2}$.

Solving Higher Degree Equations

Example 4

Given that the equation $x^3 + kx^2 - 3x + 18 = 0$ has 2 equal real roots, determine the value of k and all roots of this equation.

Solution

The only real root of this equation is $p = 3$.
We can then substitute this value into (3) to find r.

$$\begin{aligned} r &= -\frac{18}{p^2} && (3) \\ &= -\frac{18}{3^2} \\ &= -2 \end{aligned}$$

Substitute the values for p and r into (1) to find k.

$$\begin{aligned} k &= -(2p + r) && (1) \\ &= -(2(3) + (-2)) \\ &= -4 \end{aligned}$$

Therefore, $k = -4$ and the roots of the equation, $x^3 - 4x^2 - 3x + 18 = 0$, are -2 (multiplicity 1) and 3 (multiplicity 2).

Summary

- If a polynomial equation, $P(x) = 0$, is factorable, the roots of the equation are determined by factoring the polynomial, setting each factor to zero, and then solving each of these equations individually.
- An nth degree polynomial equation has at most n distinct roots.
- The solutions to a polynomial equation $P(x) = 0$ are the zeros of the corresponding polynomial function $y = P(x)$.
- The x-intercepts of the graph of $y = P(x)$ are the real zeros of the polynomial function.
- If a polynomial equation of degree 3 or greater cannot be factored, the roots of the equation must be determined using technology or higher-level mathematical procedures.