The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

Vertical Asymptotes And Discontinuity

In This Module

- We will identify the vertical asymptotes and/or point(s) of discontinuity of a rational function.
- We will study the behaviour of the rational function to the left and right side of the discontinuity.
- We will introduce the language of limits to assist us in communicating our understanding of the behaviour of the function close to its vertical asymptotes.

Vertical Asymptotes

A rational function $y = \dfrac{g(x)}{h(x)}$, $h(x) \neq 0$ will have a **vertical asymptote** at $x = a$ if $h(a) = 0$ and $g(a) \neq 0$, when the function is in simplest form.

Vertical Asymptotes

Example 1

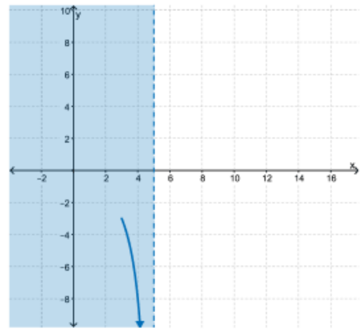
Consider the function $f(x) = \dfrac{2x}{x-5}$.

The domain of the function is $\{x \mid x \neq 5,\ x \in \mathbb{R}\}$.

Observe that $f(5) = \dfrac{2(5)}{5-5} = \dfrac{10}{0}$ which is an undefined value. The graph of the function is discontinuous at $x = 5$.

What happens to the value of the function (the value of y) as the value of $x \to 5$?

x	$f(x)$
4	-8
4.5	-18
4.9	-98
4.99	-998
4.999	-9998



This table shows, as x approaches 5 from the left, that is from numbers less than 5, y approaches a large negative value ($y \to -\infty$).

Vertical Asymptotes

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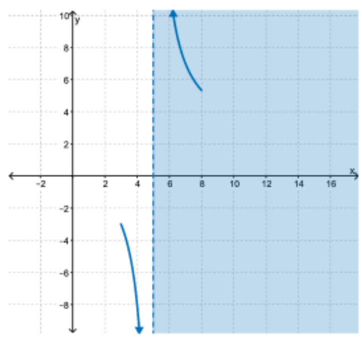
The domain of the function is $\{x \mid x \neq 5,\ x \in \mathbb{R}\}$.

Observe that $f(5) = \dfrac{2(5)}{5-5} = \dfrac{10}{0}$ which is an undefined value. The graph of the function is discontinuous at $x = 5$.

x	$f(x)$
6	12
5.5	22
5.1	102
5.01	1002
5.001	10 002



This table shows, as x approaches 5 from the right, that is from numbers greater than 5, y approaches a large positive value ($y \to \infty$).

Vertical Asymptotes

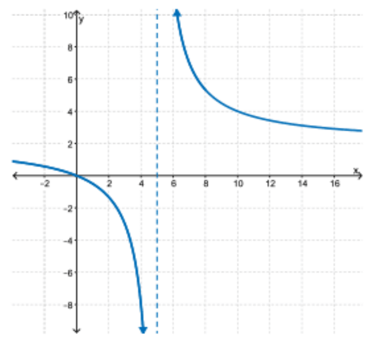
Example 1

Consider the function $f(x) = \frac{2x}{x-5}$.

The domain of the function is $\{x \mid x \neq 5,\ x \in \mathbb{R}\}$.

Observe that $f(5) = \frac{2(5)}{5-5} = \frac{10}{0}$ which is an undefined value. The graph of the function is discontinuous at $x = 5$.

The function $f(x) = \frac{2x}{x-5}$ has a vertical asymptote of $x = 5$. The graph of the function will approach, but never touch this line.



The Language of Limits

Concept	Notation
x approaches 5	$x \to 5$
x approaches 5 from the left	$x \to 5^-$ (think 5 substract a small amount)
x approaches 5 from the right	$x \to 5^+$ (think 5 plus a small amount)

When $x \to 5^-, y \to -\infty$

x	$f(x)$
4	-8
4.5	-18
4.9	-98
4.99	-998
4.999	-9998

$\lim_{x \to 5^-} f(x) = -\infty$

This means that y will continue to decrease in value as x gets closer and closer to 5 from the left of 5.

The Language of Limits

Concept	Notation
x approaches 5	$x \to 5$
x approaches 5 from the left	$x \to 5^-$ (think 5 substract a small amount)
x approaches 5 from the right	$x \to 5^+$ (think 5 plus a small amount)

When $x \to 5^+, y \to \infty$

x	$f(x)$
6	12
5.5	22
5.1	102
5.01	1002
5.001	$10\,002$

$$\lim_{x \to 5^+} f(x) = \infty$$

This means that y will continue to increase in value as x gets closer and closer to 5 from the right of 5.

The Language of Limits

The concept of a limit, which is fundamental to calculus, is used when mathematicians are concerned with the behaviour of a function near a particular value of x (or as $x \to \pm\infty$).

If the limit of a function, $y = f(x)$, as x approaches a is equal to L, then we write

$$\lim_{x \to a} f(x) = L$$

This means that $f(x)$ gets closer and closer to the value L, as x gets closer and closer to the value a. That is, $y \to L$ as $x \to a$.

A limit provides information about how a function behaves **near**, not **at**, a specific value of x.

It should also be mentioned that the $\lim_{x \to a} f(x)$ exists only when the left and right side limits exist and are equal.

In the first example, the left and right side limits, as $x \to 5$, do not exist.

For these limits to exist, $f(x)$ (that is, y) must approach a specific finite value when x approaches 5 from the left or right. Since

$$\lim_{x \to 5^-} \frac{2x}{x-5} = -\infty \text{ and } \lim_{x \to 5^+} \frac{2x}{x-5} = \infty$$

y continues to grow larger in a negative or positive direction.

It has unbounded behaviour. This type of behaviour happens at a vertical asymptote.

The $\lim_{x \to 5} \frac{2x}{x-5}$ does not exist; however, the left and right side limits have allowed us to communicate the behaviour of the function at the vertical asymptote.

Examples

Example 2

Determine the vertical asymptotes, if any, for the function $f(x) = \dfrac{-2x+4}{x^2-x-2}$ and discuss the behaviour of the function near these asymptotes.

Solution

We start by factoring the denominator of the function to identify any restrictions on the value of x. By factoring the denominator, we can determine the restrictions on x.

$$f(x) = \frac{-2x+4}{(x+1)(x-2)}$$

The domain of the function is $\{x \mid x \neq -1, 2,\ x \in \mathbb{R}\}$.
$f(-1) = \dfrac{6}{0}$, an undefined value. This indicates that there is a vertical asymptote at $x = -1$.

Examples

Example 2

Determine the vertical asymptotes, if any, for the function $f(x) = \dfrac{-2x+4}{x^2-x-2}$ and discuss the behaviour of the function near these asymptotes.

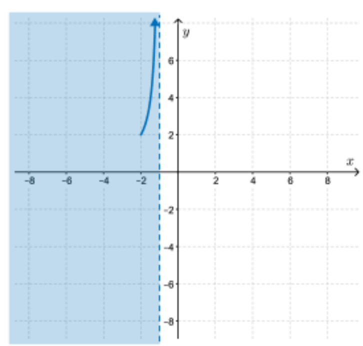
Solution

What happens to the value of $f(x)$ as $x \to -1^-$?

x	$f(x)$
-2	2
-1.5	4
-1.1	20
-1.01	200
-1.001	2000

We can see from this table that $y \to \infty$ as $x \to -1^-$.
Therefore,

$$\lim_{x \to -1^-} f(x) = \infty$$



Examples

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Determine the vertical asymptotes, if any, for the function $f(x) = \dfrac{-2x+4}{x^2-x-2}$ and discuss the behaviour of the function near these asymptotes.

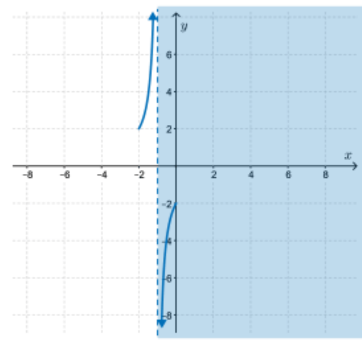
Solution

What happens to the value of $f(x)$ as $x \to -1^+$?

x	$f(x)$
0	-2
-0.5	-4
-0.9	-20
-0.99	-200
-0.999	-2000

Similarly $y \to -\infty$ as $x \to -1^+$, so

$$\lim_{x \to -1^+} f(x) = -\infty$$



Examples

Example 2

Determine the vertical asymptotes, if any, for the function $f(x) = \dfrac{-2x+4}{x^2-x-2}$ and discuss the behaviour of the function near these asymptotes.

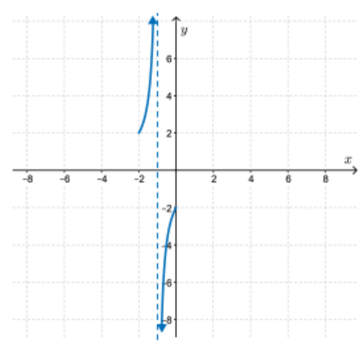
Solution

Thus,

$$\lim_{x \to -1^-} \frac{-2x+4}{x^2-x-2} = \infty \text{ and } \lim_{x \to -1^+} \frac{-2x+4}{x^2-x-2} = -\infty$$

So the limit $\lim_{x \to -1} \dfrac{-2x+4}{x^2-x-2}$ does not exist.

This unbounded behaviour of the function, to the left and right of -1, supports the fact that a vertical asymptote occurs at $x = -1$.



Examples

Example 2

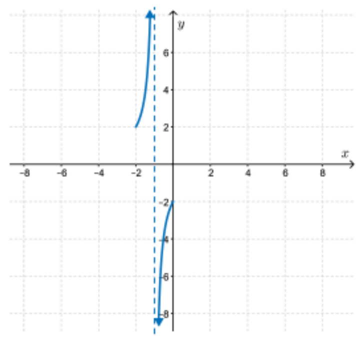
Determine the vertical asymptotes, if any, for the function $f(x) = \dfrac{-2x+4}{x^2-x-2}$ and discuss the behaviour of the function near these asymptotes.

Solution

What happens at $x = 2$?

Note that $f(2) = \dfrac{-2(2)+4}{2^2-2-2} = \dfrac{0}{0}$. This value $\dfrac{0}{0}$ is said to be indeterminate, as opposed to undefined.

Since both the numerator and denominator are 0 at $x = 2$, $x - 2$ must be a factor of both the numerator and denominator (by factor theorem). Therefore, the equation of the function can be simplified.



Examples

Example 2

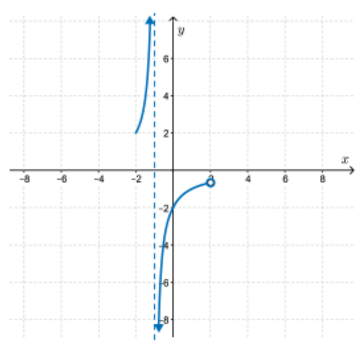
Determine the vertical asymptotes, if any, for the function $f(x) = \dfrac{-2x+4}{x^2-x-2}$ and discuss the behaviour of the function near these asymptotes.

Solution

What happens at $x = 2$?

We first study the behaviour of the function $f(x) = \dfrac{-2x+4}{x^2-x-2}$ as $x \to 2$.

x	$f(x)$
1	-1
1.5	-0.8
1.9	-0.688965
1.99	-0.6688963
1.999	-0.666889



We see that $y \to -0.\bar{6}$ or $-\dfrac{2}{3}$ as $x \to 2^-$, that is,

$$\lim_{x \to 2^-} f(x) = -\frac{2}{3}$$

Examples

Example 2

Determine the vertical asymptotes, if any, for the function $f(x) = \dfrac{-2x+4}{x^2-x-2}$ and discuss the behaviour of the function near these asymptotes.

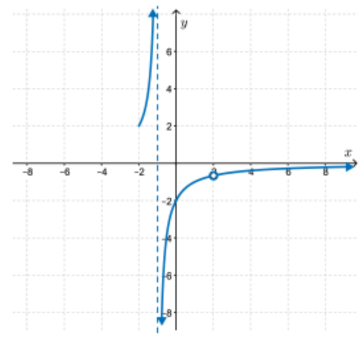
Solution

What happens at $x = 2$?

x	$f(x)$
3	-0.5
2.5	-0.5714286
2.1	-0.6451613
2.01	-0.6644518
2.001	-0.6664445

Similarly, $y \to -0.\bar{6}$ or $-\dfrac{2}{3}$ as $x \to 2^+$, that is,

$$\lim_{x\to 2^+} f(x) = -\frac{2}{3}$$



Examples

Example 2

Determine the vertical asymptotes, if any, for the function $f(x) = \dfrac{-2x+4}{x^2-x-2}$ and discuss the behaviour of the function near these asymptotes.

Solution

What happens at $x = 2$?

Thus, the left and right hand limits exist and are equal. So,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = -\frac{2}{3}$$

So the limit of $f(x)$ as $x \to 2$ exists and is $-\dfrac{2}{3}$. Therefore,

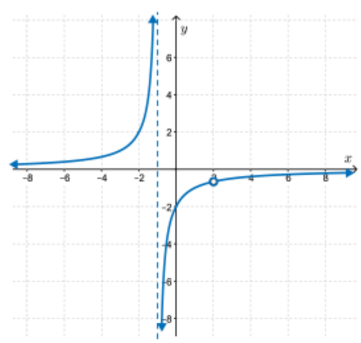
$$\lim_{x\to 2} \frac{-2x+4}{(x+1)(x-2)} = -\frac{2}{3}$$

This indicates that there is a **point of discontinuity** (a hole) at $x = 2$, and not a vertical asymptote.

The curve will approach $\left(2, -\frac{2}{3}\right)$ as the value of x approaches 2.

However, the function is not defined at $x = 2$.

An open point on the graph is used to indicate the discontinuity at $x = 2$.



Examples

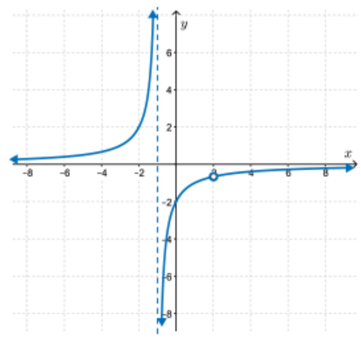
Example 2

Determine the vertical asymptotes, if any, for the function $f(x) = \dfrac{-2x+4}{x^2-x-2}$ and discuss the behaviour of the function near these asymptotes.

Solution

Algebraically, the equation of the function simplifies.

$$\begin{aligned} f(x) &= \frac{-2x+4}{x^2-x-2} \\ &= \frac{-2(x-2)}{(x+1)(x-2)} \\ f(x) &= \frac{-2}{x+1}, \ x \neq -1, 2 \end{aligned}$$



The graph of $f(x)$ is the same as the graph $y = \dfrac{-2}{x+1}$ everywhere except at the point $\left(2, -\frac{2}{3}\right)$, where $f(x)$ has hole.

Note that using the simplified equation and ignoring the restriction $x \neq 2$, $f(2) = \dfrac{-2}{2+1} = -\dfrac{2}{3}$, the y-coordinate of the point of discontinuity.

Examples

Example 3

Determine, with support, an equation for each rational function of the form $y = \dfrac{g(x)}{h(x)}$ that satisfy the given conditions.

a. A hole exists at $(4, -2)$ and a vertical asymptote occurs at $x = 0$.

b. $g(-3) = 0$ and $h(-3) = 0$, but a vertical asymptote occurs at $x = -3$.

Solution

a. For a hole to exist at $x = 4$, we need $g(4) = 0$ and $h(4) = 0$, so $x - 4$ is a factor of both the numerator and the denominator.

For a vertical asymptote to exist at $x = 0$, then $h(0) = 0$ and $g(0) \neq 0$, so x is a factor of the denominator, but not the numerator.

A function which satisfies these conditions is of the form

$$f(x) = \frac{k(x-4)}{x(x-4)}, \text{ where } k \neq 0, \ k \in \mathbb{R}, \text{ and } x \neq 0, \ 4$$

Examples

Example 3

Determine, with support, an equation for each rational function of the form $y = \dfrac{g(x)}{h(x)}$ that satisfy the given conditions.

a. A hole exists at $(4, -2)$ and a vertical asymptote occurs at $x = 0$.
b. $g(-3) = 0$ and $h(-3) = 0$, but a vertical asymptote occurs at $x = -3$.

Solution

We must determine the value of k such that the hole is located at $(4, -2)$

$$f(x) = \frac{k(x-4)}{x(x-4)} = \frac{k}{x}\,,\ x \neq 0, 4,\ k \in \mathbb{R}$$

The graph of the function, $g(x) = \dfrac{k}{x}$, $x \neq 0$ is identical to the graph of $f(x)$, with the exception of the hole at $(4, -2)$.
Using $g(4) = -2$, we can determine the value of k that will place the hole at the correct location.
Here, $\dfrac{k}{4} = -2$ and $k = -8$. Thus,

$$f(x) = \frac{-8(x-4)}{x(x-4)}\,,\ x \neq 0, 4$$

Therefore, a function with a hole at $(4, -2)$ and vertical asymptote of $x = 0$ is $f(x) = \dfrac{-8x + 32}{x^2 - 4x}$.

Examples

Example 3

Determine, with support, an equation for each rational function of the form $y = \dfrac{g(x)}{h(x)}$ that satisfy the given conditions.

a. A hole exists at $(4, -2)$ and a vertical asymptote occurs at $x = 0$.
b. $g(-3) = 0$ and $h(-3) = 0$, but a vertical asymptote occurs at $x = -3$.

Solution

b. Given $g(-3) = 0$ and $h(-3) = 0$, then $x + 3$ must be a factor of the numerator and denominator.
However, a vertical asymptote exists at $x = -3$.
This means that $h(-3) = 0$ and $g(-3) \neq 0$ when the function is in simplest form.
For this to happen, $x + 3$ must be a factor of multiplicity 2 or greater in the denominator (at least two factors of $x + 3$), and of lesser multiplicity (but at least 1) in the numerator.

One such function is given by $f(x) = \dfrac{x+3}{(x+3)^2} = \dfrac{x+3}{x^2 + 6x + 9}$, $x \neq -3$.

When this equation is simplified to $y = \dfrac{1}{x+3}$, $x \neq -3$, the indeterminate form $\left(\frac{0}{0}\right)$ of the equation at $x = -3$ is lost, but the graph of the function remains the same.
This simplified equation is not a valid solution to this problem as it does not satisfy the first condition.

Examples

Example 3

Determine, with support, an equation for each rational function of the form $y = \frac{g(x)}{h(x)}$ that satisfy the given conditions.

a. A hole exists at $(4, -2)$ and a vertical asymptote occurs at $x = 0$.

b. $g(-3) = 0$ and $h(-3) = 0$, but a vertical asymptote occurs at $x = -3$.

Solution

Therefore, the function $f(x) = \frac{x+3}{x^2+6x+9}$, $x \neq 3$ is indeterminate in form at $x = -3$, but has a vertical asymptote at $x = -3$.

In both situations, other solutions can be generated by using $y = kf(x),\ k \in \mathbb{R},\ k \neq 0$.

As well, there are other, more complicated, rational functions that would satisfy the given conditions.

Summary

For a rational function $y = \frac{g(x)}{h(x)}$, $h(x) \neq 0$:

- The function will be discontinuous at $x = a$ if $h(a) = 0$.
- The function has a vertical asymptote at $x = a$ if $h(a) = 0$ and $g(a) \neq 0$, when the function is in simplest form.
- If $h(a) = 0$ and $g(a) = 0$ for some value of $a \in \mathbb{R}$, then $x - a$ is a factor of the numerator and denominator of the function, and a point of discontinuity (a hole) may occur at $x = a$. To verify this, express the function in simplified form and then determine if it generates a single point of discontinuity, or a vertical asymptote.
- Since the value of a limit provides information near, but not at, a specific value of x, limits are often used to analyze a function near its asymptotes.