Trigonometric Ratios and Special Triangles

Introduction

The Unit Circle and Trigonometric Ratios

Recall that a circle centred at the origin with radius 1 is called a unit circle and has equation $x^2 + y^2 = 1$.

Any point on both the unit circle and the terminal arm of some standard position angle θ can be written $P(\cos(\theta), \sin(\theta))$.

Two observations can be made at this point.

We know that $\tan(\theta) = \frac{y}{x}$, and, on the unit circle, $x = \cos(\theta)$ and $y = \sin(\theta)$, so $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.

This is true for all values of θ in the domain of $\tan(\theta)$, and is called a **quotient identity**.

Also, since every point is on the unit circle $x^2 + y^2 = 1$, we can substitute $x = \cos(\theta)$ and $y = \sin(\theta)$.

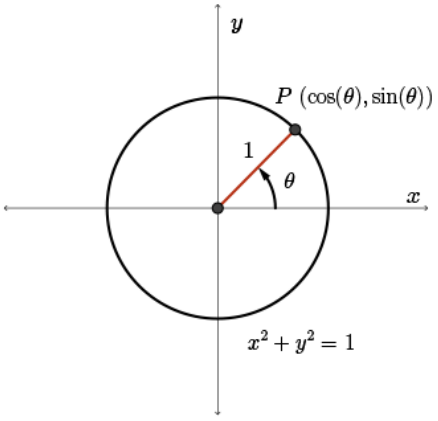
Therefore, $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$. This is usually written $\cos^2(\theta) + \sin^2(\theta) = 1$.

This relationship is true for all values of θ and is called the **Pythagorean identity**.

We will do more with identities later in this unit.

We have also defined three **reciprocal trigonometric ratios**:

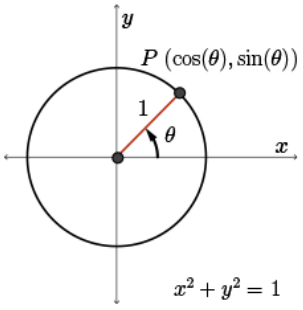
$$\csc(\theta) = \frac{1}{\sin(\theta)} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} \qquad \cot(\theta) = \frac{1}{\tan(\theta)}$$



Examples

Given any point $P(x, y)$ on the terminal arm of a standard position angle, we can determine the value of any or all of the six trigonometric ratios.

If that point is also on the unit circle, the process of determining the value of any or all of the six trigonometric ratios is easier.



Example 1

Point $P\left(-\frac{3}{5}, -\frac{4}{5}\right)$ is on the terminal arm of standard position angle θ.

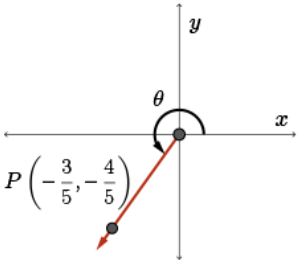
Determine the exact values of each of the six trigonometric ratios.

Solution

A sketch is generally helpful. First, we calculate the value of r.

$$r^2 = x^2 + y^2 = \left(-\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2 = \frac{9}{25} + \frac{16}{25} = 1$$

and $r = 1$ since $r > 0$. Since $r = 1$, the point $P\left(-\frac{3}{5}, -\frac{4}{5}\right)$ is on the unit circle and $(x, y) = (\cos(\theta), \sin(\theta))$. Then,



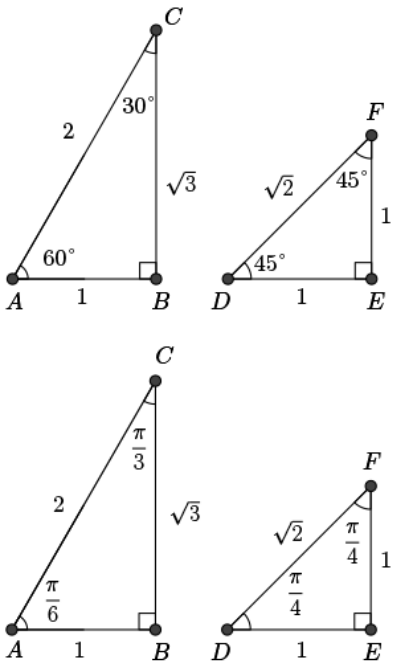
$\sin(\theta) = -\frac{4}{5}$	$\cos(\theta) = -\frac{3}{5}$	$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{4}{3}$
$\csc(\theta) = \frac{1}{\sin(\theta)} = -\frac{5}{4}$	$\sec(\theta) = \frac{1}{\cos(\theta)} = -\frac{5}{3}$	$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{3}{4}$

Special Triangles

In a previous course, two special triangles were introduced.
The first of the triangles is a $30°$-$60°$-$90°$ triangle. The ratio of the corresponding opposite side lengths is $1 : \sqrt{3} : 2$. The second triangle is a $45°$-$45°$-$90°$ triangle.
The ratio of the corresponding opposite side lengths is $1 : 1 : \sqrt{2}$.
Using radian measure, the first triangle is a $\frac{\pi}{6} - \frac{\pi}{3} - \frac{\pi}{2}$ triangle and the second triangle is a $\frac{\pi}{4} - \frac{\pi}{4} - \frac{\pi}{2}$ triangle.
The ratio of the side lengths remains the same.

Using these triangles, we are able to determine the exact value of each of the six trigonometric ratios corresponding to angles $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ or $30°$, $45°$, and $60°$.

Determine the exact values of the six trigonometric ratios for each of the angles $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$.

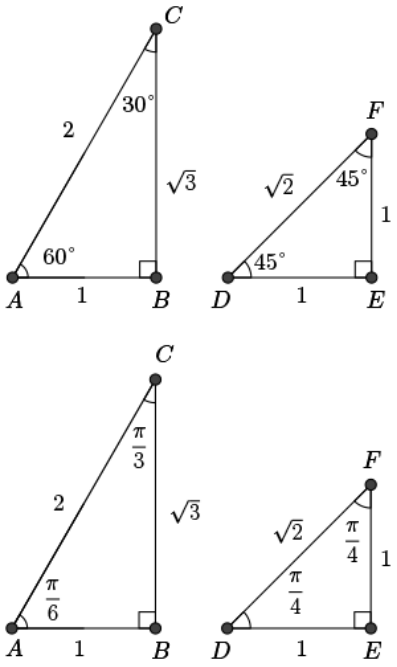


Special Triangles

Determine the exact values of the six trigonometric ratios for each of the angles $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$.

Angle (θ)	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$\frac{\pi}{6}$ or $30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$ or $45°$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$ or $60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

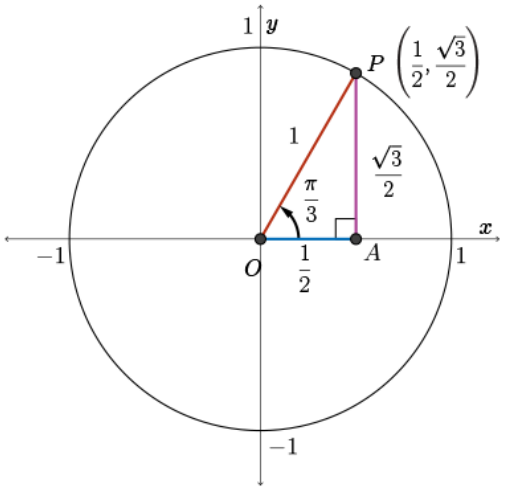
Angle (θ)	$\csc(\theta)$	$\sec(\theta)$	$\cot(\theta)$
$\frac{\pi}{6}$ or $30°$	2	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$ or $45°$	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$ or $60°$	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$	2	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$



Examples

Example 2

A unit circle is shown with OP on the terminal arm of a $\frac{\pi}{3}$ radian standard position angle. Determine the coordinates of P.



Solution

Drop a perpendicular from P to the x-axis at A.

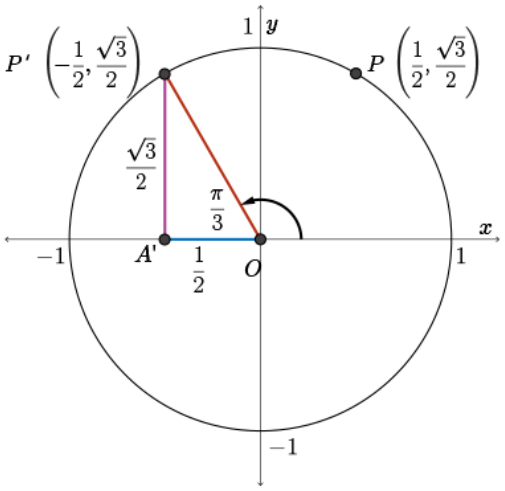
We know that $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, and $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$.

We also know that, on the unit circle, each point has coordinates $(\cos(\theta), \sin(\theta))$. Therefore, P has coordinates $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and it follows that $OA = \frac{1}{2}$, $AP = \frac{\sqrt{3}}{2}$, and $OP = 1$.

Examples

Example 2

A unit circle is shown with OP on the terminal arm of a $\frac{\pi}{3}$ radian standard position angle. Determine the coordinates of P.



Solution

Reflect $\triangle OAP$ about the y-axis, forming congruent $\triangle OA'P'$.

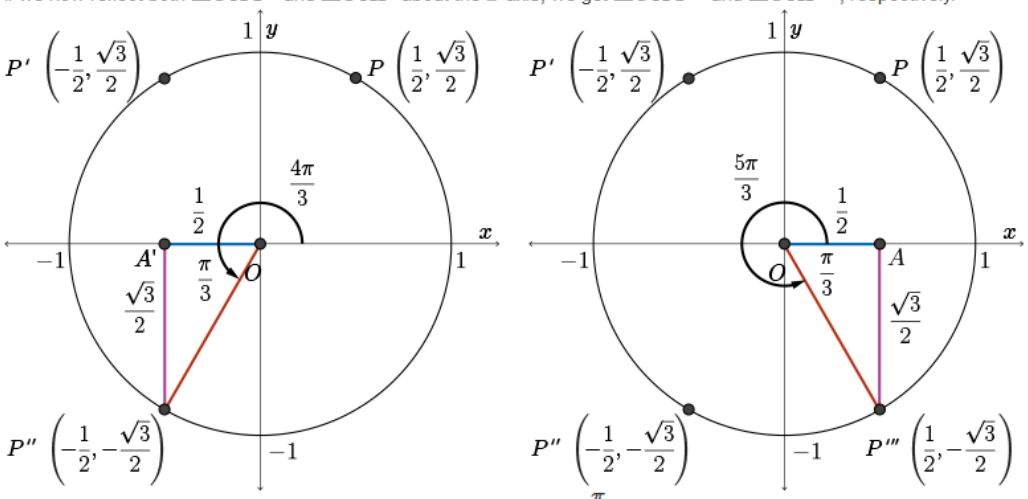
Using properties of reflections, P' has coordinates $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\angle P'OA' = \angle POA = \frac{\pi}{3}$.

Then OP' is along the terminal arm of a $\frac{2\pi}{3}$ standard position angle. It follows that

$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$, and $\tan\left(\frac{2\pi}{3}\right) = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$.

Special Triangles and the Unit Circle

If we now reflect both $\triangle OA'P'$ and $\triangle OAP$ about the x-axis, we get $\triangle OA'P''$ and $\triangle OAP'''$, respectively.



The four triangles we have seen are congruent and each contains a $\frac{\pi}{3}$ radian reference angle.

We also obtain $P''\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and $P'''\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ with points on the terminal arm of a $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$ standard position angle, respectively.

Special Triangles and the Unit Circle

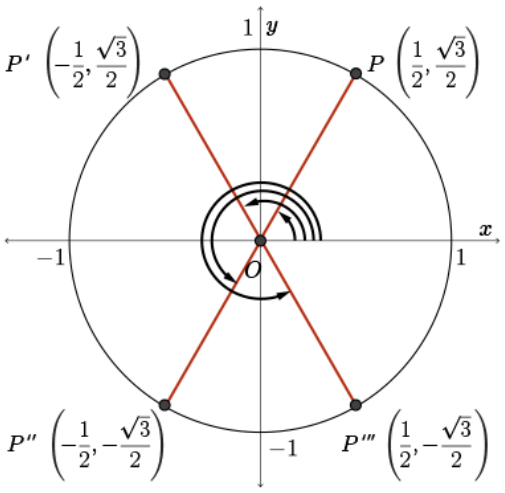
Each of the four points, P, P', P'', and P''' lie on the unit circle.

The standard position angles drawn to P, P', P'', and P''' all have the same reference angle: $\frac{\pi}{3}$ in this case.

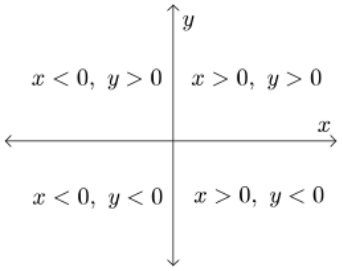
If we disregard signs, all of the four points have the same x and y-coordinates.

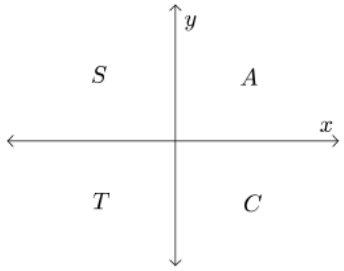
So for each of the four related standard position angles, each trigonometric ratio, without regard for the sign, will be equal.

The sign of the trigonometric ratio depends on the quadrant in which the terminal arm lies.



Determining the Sign of a Trigonometric Ratio





In previous courses, you may have determined the sign of a particular trigonometric ratio by analyzing the signs of x and y in the quadrant containing the terminal arm of the standard position angle.

Others may have used a memory device known as the **CAST rule**.

Each letter in the rule tells which trigonometric ratio is positive in a particular quadrant.
In quadrant 1, **a**ll trigonometric ratios are positive.
In quadrant 2, **s**in(θ) is positive. ($\csc(\theta)$ is also positive.)
In quadrant 3, **t**an(θ) is positive. ($\cot(\theta)$ is also positive.)
In quadrant 4, **c**os(θ) is positive. ($\sec(\theta)$ is also positive.)

Note: The trigonometric ratios associated with any reference angle are positive. (The terms related acute angle and reference angle mean the same thing. In this course, both terms are used.)

Examples

Example 3

Determine the exact value of $\sin\left(\frac{7\pi}{4}\right)$.

Solution

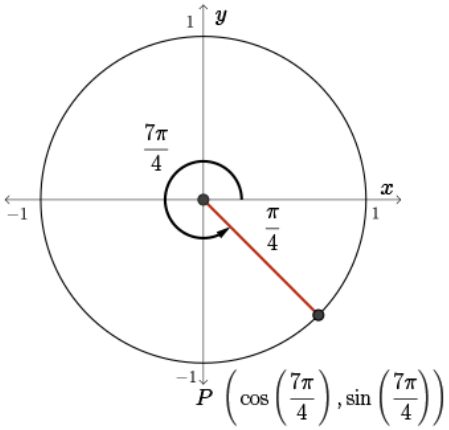
The standard position angle $\frac{7\pi}{4}$ has its terminal arm in quadrant 4.

The reference angle is $\frac{\pi}{4}$ and we know that $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

The sine function is negative in quadrant 4.

Therefore, $\sin\left(\frac{7\pi}{4}\right) = \sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

This is the exact value of $\sin\left(\frac{7\pi}{4}\right)$.



Most calculators will be able to give the approximate value of a trig ratio but not the exact value.

When $-\frac{1}{\sqrt{2}}$ is converted to a decimal, it is -0.707, rounded to three decimals.

This second answer is an approximate answer.

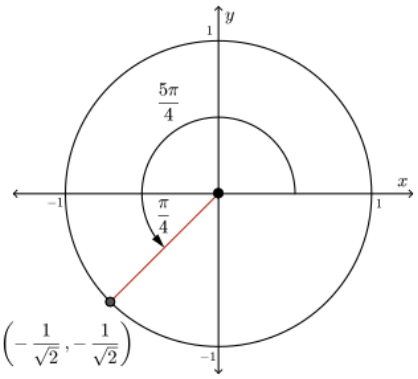
The Reference Angle $\frac{\pi}{4}$

Since $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is on the unit circle and on the terminal arm of a standard position angle $\frac{\pi}{4}$.

The point $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ is also on the unit circle and would be on the terminal arm of a standard position angle, $\frac{7\pi}{4}$.

Using symmetry arguments, we also get the point $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is on the unit circle corresponding to a standard position angle, $\frac{3\pi}{4}$.

The point $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ is on the unit circle corresponding to a standard position angle, $\frac{5\pi}{4}$.



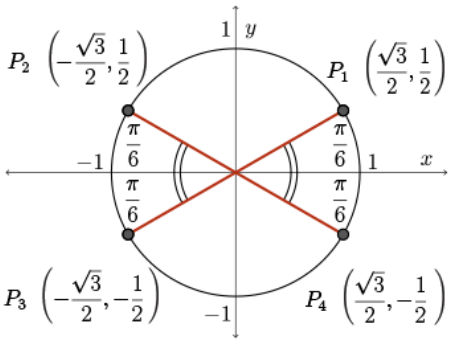
The Reference Angle $\frac{\pi}{6}$

We looked at the exact trigonometric ratios for angles between 0 and 2π with reference angles $\frac{\pi}{3}$ and $\frac{\pi}{4}$.

We have also determined the corresponding points on the unit circle.

Determine the coordinates of the four points on the unit circle corresponding to angles between 0 and 2π associated with reference angle $\frac{\pi}{6}$. Since $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, the point $P_1\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is on the unit circle and the terminal arm of standard position angle $\frac{\pi}{6}$. The other three points are easily found. The four points are shown in the chart with the corresponding standard position angle. The points are also illustrated on the diagram.

Quadrant the Terminal Arm is in	Standard Position Angle	Point on Unit Circle
1	$\frac{\pi}{6}$	$P_1\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
2	$\pi - \frac{\pi}{6} = \frac{5\pi}{6}$	$P_2\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
3	$\pi + \frac{\pi}{6} = \frac{7\pi}{6}$	$P_3\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
4	$2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$	$P_4\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$



Examples

Example 4

Without using a calculator, determine the exact value of $\cot(\pi)$.

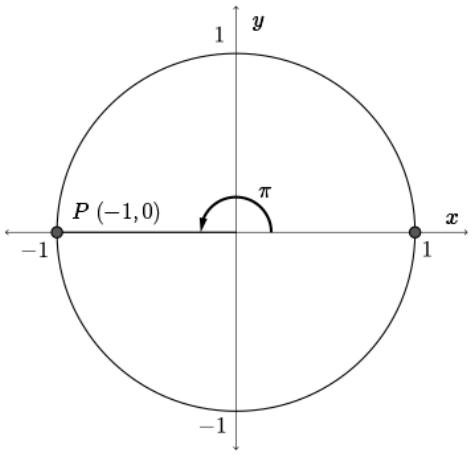
Solution

The point $(-1, 0)$ is on the unit circle and the terminal arm of the standard position angle of π radians.

We know that $\tan(\theta) = \frac{y}{x}$ so $\tan(\pi) = \frac{0}{-1}$.

Therefore, $\cot(\pi) = -\frac{-1}{0}$ and is undefined.

For any integer multiple of $\frac{\pi}{2}$ radians or 90°, the exact value of any trigonometric ratio can be found by selecting a point on the appropriate axis.



Examples

Example 5

If $\tan(\theta) = -\sqrt{3}$, determine the possible values of θ such that $0 \leq \theta \leq 2\pi$.

Solution

Some observations are helpful here. First, the ratio $\frac{\sqrt{3}}{1}$ is familiar. We know that $\tan\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{1}$. It follows that $\frac{\pi}{3}$ is the reference angle.

Second, $\tan(\theta) < 0$. Therefore, the terminal arm can be in quadrant 2 or 4.

That is, there are two possible answers in the required domain.

In quadrant 2, $\theta_1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

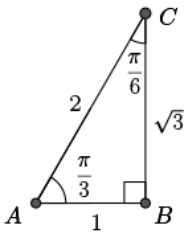
In quadrant 4, $\theta_2 = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

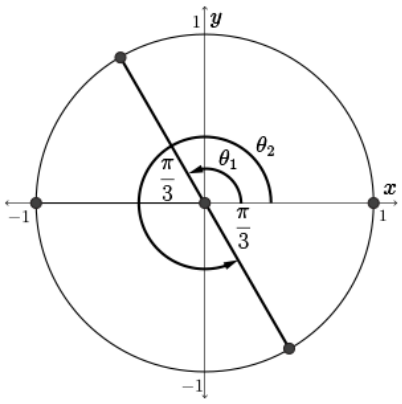
If there had been no restriction on the domain, we would have to add integer multiples of 2π to the answers.

That is, $\theta = \left\{\frac{2\pi}{3} + 2\pi n, \frac{5\pi}{3} + 2\pi n, n \in \mathbb{Z}\right\}$.

Check to see if any coterminal angles are in the given domain.

In this case, there are no additional coterminal angles.

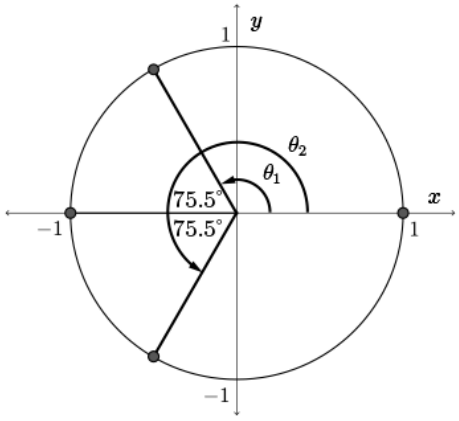




Examples

Example 6

If $\cos(\theta) = -\frac{1}{4}$, determine the possible values of θ such that $-180^\circ \leq \theta \leq 180^\circ$.



Solution

Some observations are helpful here.

First, $\cos(\theta) < 0$. Therefore, the terminal arm can be in quadrant 2 or 3.

Second, the ratio $\frac{1}{4}$ is unfamiliar.

We can use a calculator to determine the approximate value of the reference angle.

Let α be the reference angle. Then $\cos(\alpha) = +\frac{1}{4}$.

(The trig ratio of any angle between 0° and 90° is greater than zero.)

Using your calculator, $\alpha = \cos^{-1}\left(\frac{1}{4}\right) \approx 75.5^\circ$.

Be sure your calculator is set in degree mode for this example.

In quadrant 2, $\theta_1 = 180^\circ - \alpha \approx 104.5^\circ$. In quadrant 3, $\theta_2 = 180^\circ + \alpha \approx 255.5^\circ$.

Examples

Example 7

If $\sin(\theta) = \frac{1}{2}$, determine the possible values of the other five trigonometric ratios.

Solution

Since $\sin(\theta) > 0$, the terminal arm can be in quadrant 1 or 2. We are looking for two sets of trigonometric ratios.

Second, if we use the unit circle, the points have coordinates $(\cos(\theta), \sin(\theta)) = \left(\cos(\theta), \frac{1}{2}\right)$.

We know that $x^2 + y^2 = 1$, so $\cos^2(\theta) + \frac{1}{4} = 1$ and $\cos^2(\theta) = \frac{3}{4}$ follows.

Therefore, $\cos(\theta) = \pm\frac{\sqrt{3}}{2}$. The two points on the unit circle are $P_1\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $P_2\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Using the definitions with $P_1\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$,

$$\sin(\theta) = \frac{1}{2}, \cos(\theta) = \frac{\sqrt{3}}{2}, \tan(\theta) = \frac{1}{\sqrt{3}}, \csc(\theta) = 2, \sec(\theta) = \frac{2}{\sqrt{3}}, \cot(\theta) = \sqrt{3}$$

Using the definitions with $P_2\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$,

$$\sin(\theta) = \frac{1}{2}, \cos(\theta) = -\frac{\sqrt{3}}{2}, \tan(\theta) = -\frac{1}{\sqrt{3}}, \csc(\theta) = 2, \sec(\theta) = -\frac{2}{\sqrt{3}}, \cot(\theta) = -\sqrt{3}$$

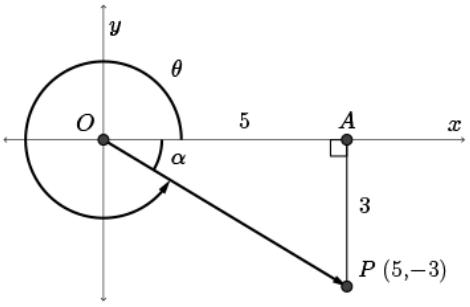
Examples

Example 8

The point $P(5, -3)$ is on the terminal arm of some standard position angle θ, such that $0 \leq \theta \leq 2\pi$. Determine the measure of θ, correct to two decimal places.

Solution

Notice that $P(5, -3)$ is a point in the fourth quadrant.
Let α represent the reference angle.
From the diagram, $OA = 5$ and $AP = 3$.
Then $\tan(\alpha) = \frac{3}{5}$ and $\alpha = \tan^{-1}\left(\frac{3}{5}\right) \approx 0.54$ radians.
It follows that $\theta = 2\pi - \alpha \approx 5.74$ radians.
It should be noted that you could go directly from the definition $\tan(\theta) = \frac{y}{x} = -\frac{3}{5}$.
Then it depends on what your calculator produces as an answer.
You must interpret the result in light of the problem.
Finding the reference angle and placing the terminal arm in the correct quadrant removes any doubt.



Examples

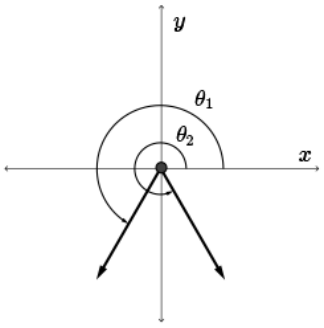
Example 9

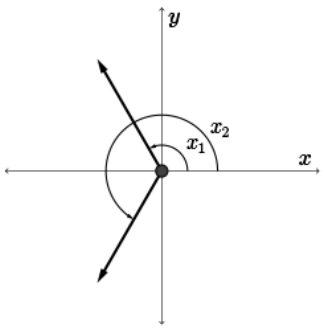
Solve $2\sin\left(x + \frac{\pi}{2}\right) = -\sqrt{3}, 0 \leq x \leq 3\pi$. State the exact answers.

Solution

Let $\theta = x + \frac{\pi}{2}$. The equation simplifies to $2\sin(\theta) = -\sqrt{3}$.
Dividing both sides by 2, we get $\sin(\theta) = -\frac{\sqrt{3}}{2}$.
Since $\sin(\theta) < 0$, the terminal arm of the standard position angle is in either quadrant 3 or quadrant 4.
If α is the reference angle, then $\sin\alpha = \frac{\sqrt{3}}{2}$ and $\alpha = \frac{\pi}{3}$.

In quadrant 3, $\theta_1 = \pi + \alpha = \frac{4\pi}{3}$.	In quadrant 4, $\theta_2 = 2\pi - \alpha = \frac{5\pi}{3}$.
But $\theta = x + \frac{\pi}{2}$	But $\theta = x + \frac{\pi}{2}$
$x_1 = \theta_1 - \frac{\pi}{2}$	$x_2 = \theta_2 - \frac{\pi}{2}$
$x_1 = \frac{4\pi}{3} - \frac{\pi}{2}$	$x_2 = \frac{5\pi}{3} - \frac{\pi}{2}$
$x_1 = \frac{8\pi}{6} - \frac{3\pi}{6}$	$x_2 = \frac{10\pi}{6} - \frac{3\pi}{6}$
$x_1 = \frac{5\pi}{6}$	$x_2 = \frac{7\pi}{6}$

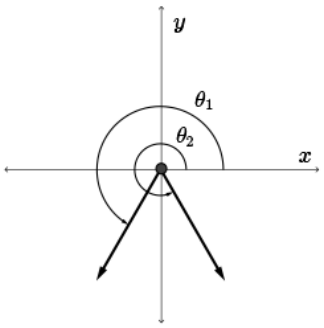




Examples

Example 9

Solve $2\sin\left(x+\frac{\pi}{2}\right)=-\sqrt{3}, 0\leq x\leq 3\pi$. State the exact answers.



Solution

Both $x=\frac{5\pi}{6}$ and $x=\frac{7\pi}{6}$ are in the domain $0\leq x\leq 3\pi$.

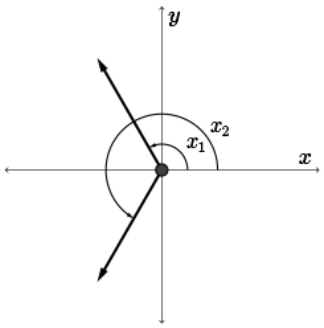
We must check for coterminal angles in the domain.

Subtracting 2π from either angle produces a negative angle outside of the domain.

However, $\frac{5\pi}{6}+2\pi=\frac{17\pi}{6}$, which is in the domain.

Adding 2π to either $\frac{17\pi}{6}$ or $\frac{7\pi}{6}$ produces an angle outside of the domain.

Therefore, $x=\left\{\frac{5\pi}{6},\frac{7\pi}{6},\frac{17\pi}{6}\right\}$.



Summary

In this module,

- we determined the exact and approximate values of various trigonometric ratios, and
- we worked with special triangles and solved a variety of equations.

In future modules, we will use methods developed in this module to assist in the graphing of trigonometric functions and in the solving of more difficult applications.