The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

Curve Sketching

In This Module

- We will revisit the algorithm for curve sketching and apply it to curves whose equations involve exponential, logarithmic, and trigonometric functions.

Algorithm for Curve Sketching

Here is a quick review of the steps:

1. Domain
2. y-intercept
3. Discontinuities
4. x-intercept(s)
5. Horizontal asymptote(s)
6. Oblique asymptote(s)
7. Critical point(s) (where $f' = 0$ or DNE)
8. Intervals of increase/decrease
9. Possible point(s) of inflection (where $f'' = 0$ or DNE)
10. Intervals of concavity
11. Sketch

Since we are no longer dealing with only polynomials and rational functions, we may need to use l'Hospital's rule to aid in computing limits.

Examples

Example 1

Sketch the graph of the function $y = xe^{-x}$.

Solution

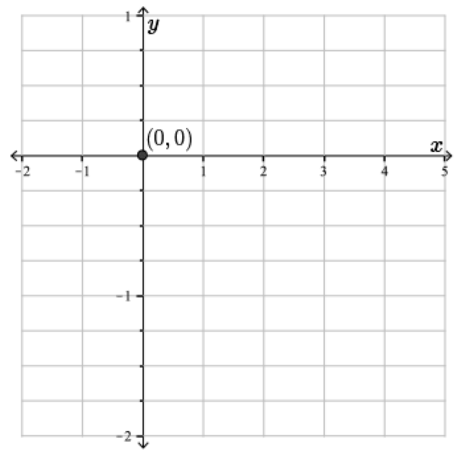
Let $f(x) = xe^{-x}$.

1. The function $f(x)$ is a product of a polynomial and an exponential which both have domain $\mathbb{R}$, and so the domain of $f(x)$ is $\mathbb{R}$.

2. The y-intercept of $f(x)$ is $f(0) = 0e^{-0} = 0$.

3. Since x and e^{-x} are both continuous functions, so is $f(x)$. Therefore, $f(x)$ has no discontinuities.

4. Since $e^{-x} \neq 0$, we have $f(x) = xe^{-x} = 0$ if and only if $x = 0$, the x-intercept of $f(x)$ is $x = 0$.



Examples

Example 1

Sketch the graph of the function $y = xe^{-x}$.

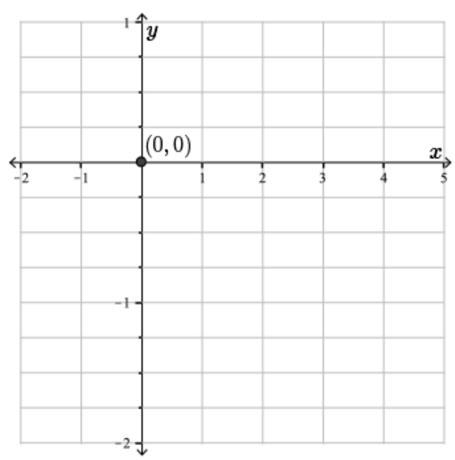
Solution

5-6. We need to examine the behaviour of the function $f(x)$ as x approaches $\pm\infty$:

Consider $\lim\limits_{x\to\infty} xe^{-x} = \lim\limits_{x\to\infty} \frac{x}{e^x}$.

Since this is an indeterminate form of type $\frac{\infty}{\infty}$, we can use l'Hospital's rule to evaluate the limit:

$$\begin{aligned}\lim_{x\to\infty} xe^{-x} &= \lim_{x\to\infty} \frac{x}{e^x} \\ &= \lim_{x\to\infty} \frac{\frac{d}{dx}\left(x\right)}{\frac{d}{dx}\left(e^x\right)} \\ &= \lim_{x\to\infty} \frac{1}{e^x} \\ &= 0\end{aligned}$$



Examples

Example 1

Sketch the graph of the function $y = xe^{-x}$.

Solution

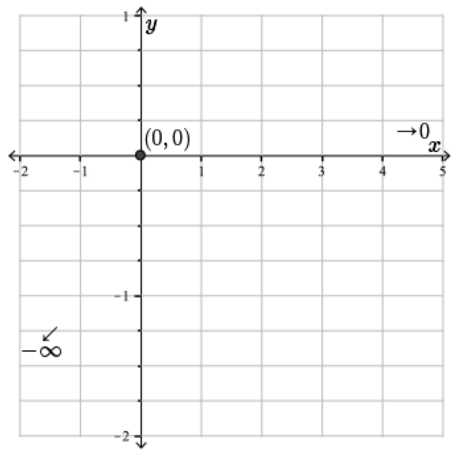
Now consider $\lim_{x \to -\infty} xe^{-x}$.

We can see from a sketch of $y = e^{-x}$ that, as $x \to -\infty$, $e^{-x} \to \infty$ and so we have

$$\lim_{x \to -\infty} xe^{-x} \to -\infty$$

Since $f(x) = xe^{-x} \geq 0$ for $x \geq 0$, the function approaches the horizontal asymptote $y = 0$ from above.

Here, we have no oblique asymptote as the function decreases exponentially as $x \to -\infty$.



Examples

Example 1

Sketch the graph of the function $y = xe^{-x}$.

Solution

7. Next we determine the critical points of $f(x)$.

Using the product rule, we get

$f'(x) = x(-e^{-x}) + e^{-x}(1) = e^{-x} - xe^{-x} = (1-x)e^{-x}$

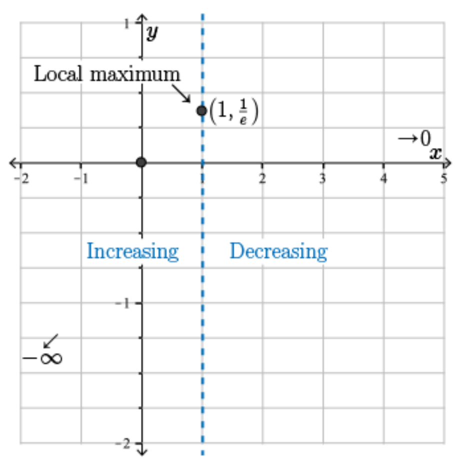
Since $f'(x)$ is defined for all real numbers x, critical points only occur when $f'(x) = 0$.

As $e^{-x} \neq 0$, we have $f'(x) = 0$ if and only if $x = 1$.

8. Now we can determine the intervals of increase and decrease using the following table:

Interval	$x < 1$	$x > 1$
$f'(x) = (1-x)e^{-x}$	$(+)(+) > 0$	$(-)(+) < 0$
$f(x)$	Increasing	Decreasing

Therefore, $x = 1$ is a local *maximum*, and the maximum value is $f(1) = 1e^{-1} = \frac{1}{e} \approx 0.37$.



Examples

Example 1

Sketch the graph of the function $y = xe^{-x}$.

Solution

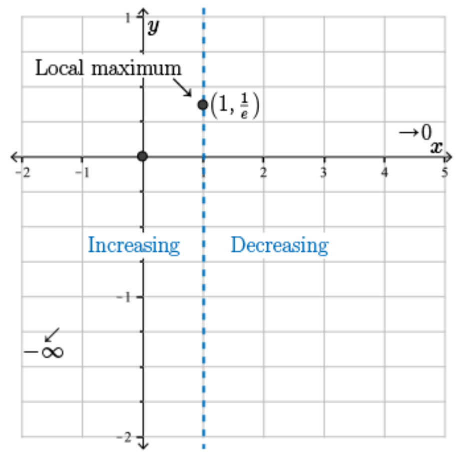
9. Finally, we find all points at which $f(x)$ may change concavity. A concavity change may occur where $f''(x) = 0$ or where $f''(x)$ does not exist.

We have $f'(x) = (1-x)e^{-x} = e^{-x} - xe^{-x}$ and so, using the product rule, we have

$$\begin{aligned} f''(x) &= -e^{-x} - \left[x(-e^{-x}) + e^{-x}\right] \\ &= -2e^{-x} + xe^{-x} \\ &= (x-2)e^{-x} \end{aligned}$$

Therefore, $f''(x) = 0$ if and only if $x = 2$. Note that $f''(x)$ is defined for all x.

The function value at this point is $f(2) = 2e^{-2} \approx 0.27$.



Examples

Example 1

Sketch the graph of the function $y = xe^{-x}$.

Solution

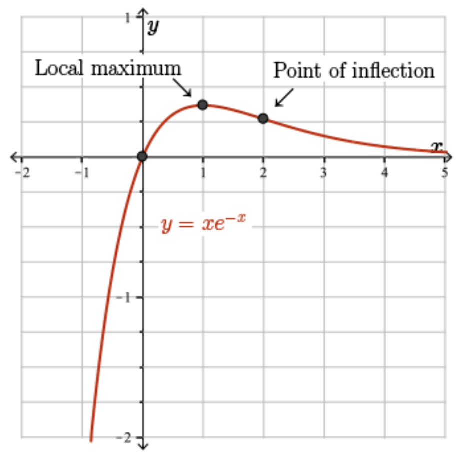
10. We can now determine the intervals on which the function is concave up and concave down.

Since the domain of $f''(x)$ is all of $\mathbb{R}$, the concavity of $f(x)$ can only change at the point of inflection $x = 2$.

Interval	$x < 2$	$x > 2$
$f''(x) = (x-2)e^{-x}$	$(-)(+) < 0$	$(+)(+) > 0$
$f(x)$	Concave down	Concave up

Therefore, $f(x)$ has a point of inflection at $(2, 2e^{-2})$.

11. Using all of the information that we have gathered, we make the following sketch.



Examples

Example 2

Sketch the graph of the function $y = \dfrac{\cos(x)}{1+\sin(x)}$.

Solution

Let $f(x) = \dfrac{\cos(x)}{1+\sin(x)}$.

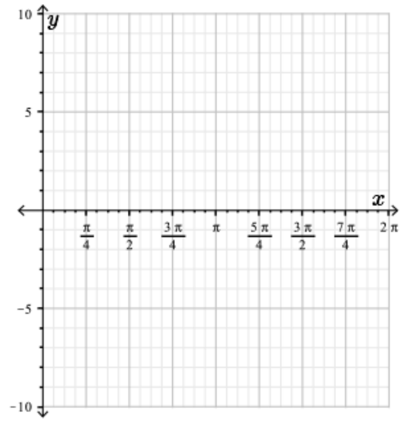
Since $\cos(x)$ and $1+\sin(x)$ have period 2π, so does $f(x)$.
We will sketch the curve $y = f(x)$ over the interval $[0, 2\pi]$.

1. The functions $\cos(x)$ and $1+\sin(x)$ are both continuous, and so the domain of $f(x)$ is all real numbers x such that $1+\sin(x) \neq 0$.

We have $\sin(x) = -1$ when $x = \dfrac{3\pi}{2} + 2k\pi$, for any integer k, and so the domain of $f(x)$ is

$$\left\{x \in \mathbb{R} : x \neq \tfrac{3\pi}{2} + 2k\pi, k \text{ is any integer }\right\}$$

On the interval $[0, 2\pi]$, $f(x)$ is only undefined at the point $x = \dfrac{3\pi}{2}$.



Examples

Example 2

Sketch the graph of the function $y = \dfrac{\cos(x)}{1+\sin(x)}$.

Solution

2. The y-intercept of $f(x)$ is

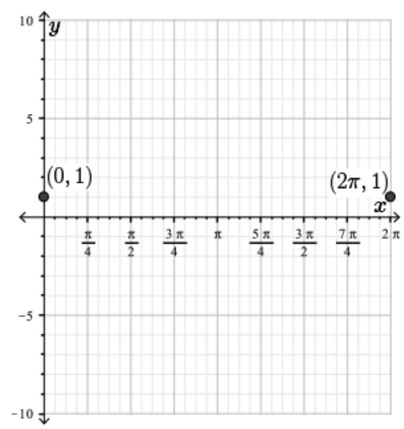
$$f(0) = \frac{\cos(0)}{1+\sin(0)} = \frac{1}{1+0} = 1$$

Since $f(x)$ has period 2π, we also have $f(2\pi) = 1$.

3. The function $f(x)$ has a discontinuity at each point where $\sin(x) = -1$, that is at $x = \dfrac{3\pi}{2} + 2k\pi$ for any integer k.

There is one point of discontinuity in the interval $[0, 2\pi]$ at $x = \dfrac{3\pi}{2}$.

Let's examine the behaviour of the function as it approaches this discontinuity from the left and from the right.



Examples

Example 2

Sketch the graph of the function $y = \dfrac{\cos(x)}{1+\sin(x)}$.

Solution

As $\cos\left(\dfrac{3\pi}{2}\right) = 0$ and $1 + \sin\left(\dfrac{3\pi}{2}\right) = 1 + (-1) = 0$, the limits as $x \to \dfrac{3\pi}{2}^-$ and $x \to \dfrac{3\pi}{2}^+$ are indeterminate forms of type $\dfrac{0}{0}$.

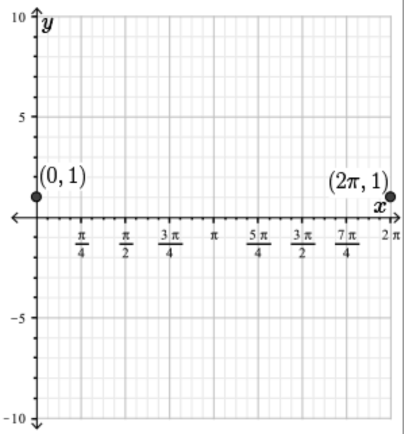
Using L'Hospital's rule we have

$$\lim_{x\to\frac{3\pi}{2}^-} \frac{\cos(x)}{1+\sin(x)} = \lim_{x\to\frac{3\pi}{2}^-} \frac{\frac{d}{dx}\Big(\cos(x)\Big)}{\frac{d}{dx}\Big(1+\sin(x)\Big)} = \lim_{x\to\frac{3\pi}{2}^-} \frac{-\sin(x)}{\cos(x)}$$

provided that the second limit exists or approaches $\pm\infty$.

As $x \to \dfrac{3\pi}{2}^-$, $\dfrac{-\sin(x)}{\cos(x)} \approx \dfrac{-(-1)}{0^-} \to -\infty$ and so

$$\lim_{x\to\frac{3\pi}{2}^-} \frac{\cos(x)}{1+\sin(x)} \to -\infty$$



Examples

Example 2

Sketch the graph of the function $y = \dfrac{\cos(x)}{1+\sin(x)}$.

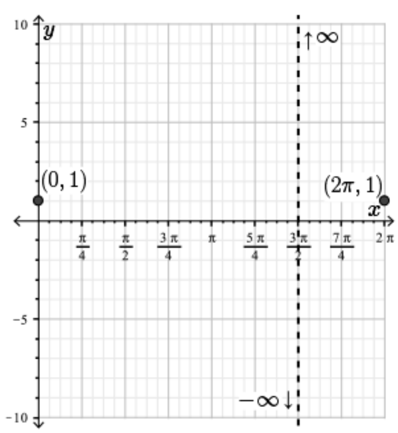
Solution

Similarly, using l'Hospital's rule, we have

$$\lim_{x\to\frac{3\pi}{2}^+} \frac{\cos(x)}{1+\sin(x)} = \lim_{x\to\frac{3\pi}{2}^+} \frac{-\sin(x)}{\cos(x)}$$

and, as $x \to \dfrac{3\pi}{2}^+$, we have $\dfrac{-\sin(x)}{\cos(x)} \approx \dfrac{-(-1)}{0^+} \to +\infty$ and hence the limit is $+\infty$.

Therefore, $f(x)$ has a vertical asymptote at $x = \dfrac{3\pi}{2}$.



Examples

Example 2

Sketch the graph of the function $y = \dfrac{\cos(x)}{1 + \sin(x)}$.

Solution

4. The roots of $f(x)$ occur when the numerator of $f(x)$ is 0, but the denominator is non-zero.

Over the interval $[0, 2\pi]$, we have $\cos(x) = 0$ when $x = \dfrac{\pi}{2}$ and $x = \dfrac{3\pi}{2}$.

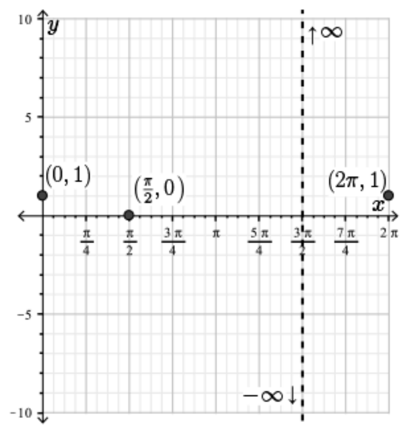
Since

$$\sin\left(\frac{3\pi}{2}\right) = -1 \Rightarrow 1 + \sin\left(\frac{3\pi}{2}\right) = 0$$

$$\sin\left(\frac{\pi}{2}\right) = 0 \Rightarrow 1 + \sin\left(\frac{\pi}{2}\right) \neq 0$$

only $x = \dfrac{\pi}{2}$ is a root of $f(x)$.

So in the interval $[0, 2\pi]$, we have one x-intercept at $x = \dfrac{\pi}{2}$.



5-6. As $f(x)$ is periodic, the limit of $f(x)$ as x approaches positive or negative infinity does not exist.

Examples

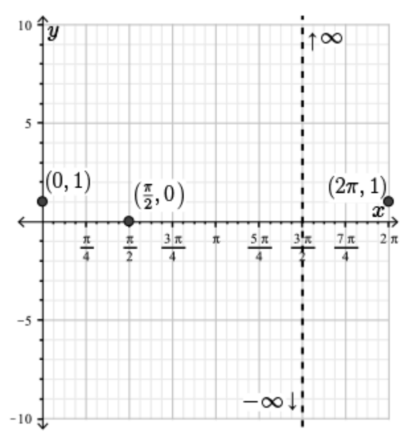
Example 2

Sketch the graph of the function $y = \dfrac{\cos(x)}{1 + \sin(x)}$.

Solution

7. We now find $f'(x)$ using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(1+\sin(x))\dfrac{d}{dx}\left[\cos(x)\right] - \cos(x)\dfrac{d}{dx}\left[1+\sin(x)\right]}{(1+\sin(x))^2} \\ &= \frac{(1+\sin(x))[-\sin(x)] - \cos(x)[\cos(x)]}{(1+\sin(x))^2} \\ &= \frac{-\sin(x) - \sin^2(x) - \cos^2(x)}{(1+\sin(x))^2} \\ &= \frac{-\sin(x) - (\sin^2(x) + \cos^2(x))}{(1+\sin(x))^2} \\ &= \frac{-\sin(x) - 1}{(1+\sin(x))^2} \\ &= \frac{-(1+\sin(x))}{(1+\sin(x))^2} = -\frac{1}{1+\sin(x)} \end{aligned}$$



Examples

Example 2

Sketch the graph of the function $y = \dfrac{\cos(x)}{1+\sin(x)}$.

Solution

$$f'(x) = -\frac{1}{1+\sin(x)}$$

The derivative $f'(x)$ is undefined when $\sin(x) = -1$.

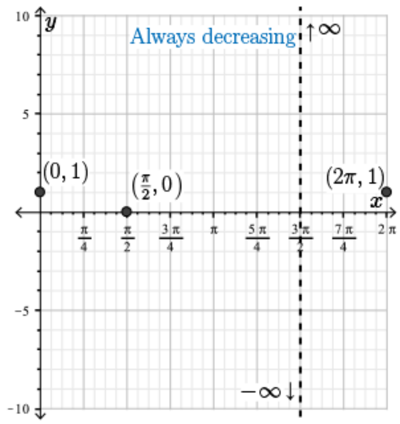
Over the interval $[0, 2\pi]$, this occurs at $x = \dfrac{3\pi}{2}$.

Although $f'(x)$ is not defined at $x = \dfrac{3\pi}{2}$, there is no critical point here since $x = \dfrac{3\pi}{2}$ does not lie in the domain of f.

In particular, the function $f(x)$ has no local extremes.

8. Since $\sin(x) \geq -1$ we have $1+\sin(x) \geq 0$ for all x and, in particular, $1+\sin(x) > 0$ for all x in the domain of $f(x)$.

Therefore, we have $f'(x) = -\dfrac{1}{1+\sin(x)} < 0$ for all x in the domain of $f(x)$ and so the function $f(x)$ is always decreasing.



Examples

Example 2

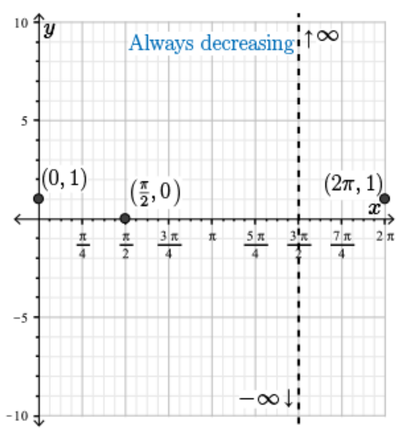
Sketch the graph of the function $y = \dfrac{\cos(x)}{1+\sin(x)}$.

Solution

9. Finally, we find all points at which $f(x)$ may change concavity. A concavity change may occur where $f''(x) = 0$ or where $f''(x)$ does not exist. Since $f'(x) = -\frac{1}{1+\sin(x)} = -(1+\sin(x))^{-1}$ we will use the power rule and the chain rule to find $f''(x)$:

$$\begin{aligned} f''(x) &= +(1+\sin(x))^{-2}\,\frac{d}{dx}\left[1+\sin(x)\right] \\ &= \frac{\cos(x)}{(1+\sin(x))^2} \end{aligned}$$

We have $f''(x) = 0$ when $\cos(x) = 0$ and $1+\sin(x) \neq 0$. This occurs when $x = \dfrac{\pi}{2}$. We have f'' undefined when $1+\sin(x) = 0$. This occurs when $x = \dfrac{3\pi}{2}$. These are the only two x values at which $f(x)$ may change concavity.



Notice that the only possible point of inflection, $\left(\frac{\pi}{2}, 0\right)$, has already been plotted on our graph.

Examples

Example 2

Sketch the graph of the function $y = \dfrac{\cos(x)}{1+\sin(x)}$.

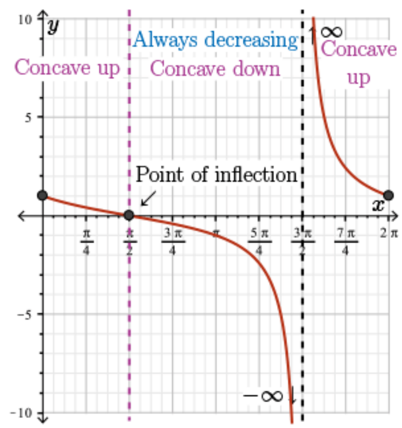
Solution

10. We can now determine the intervals on which the function is concave up and concave down.
Again, we will restrict ourselves to the interval $[0, 2\pi]$.

Interval	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
$f''(x) = \dfrac{\cos(x)}{(1+\sin(x))^2}$	$\dfrac{(+)}{(+)} > 0$	$\dfrac{(-)}{(+)} < 0$	$\dfrac{(+)}{(+)} > 0$
$f(x)$	Concave up	Concave down	Concave up

Observe that $f(x)$ has a point of inflection at $x = \dfrac{\pi}{2}$.

11. Using all of the information that we have gathered, we make the following sketch.

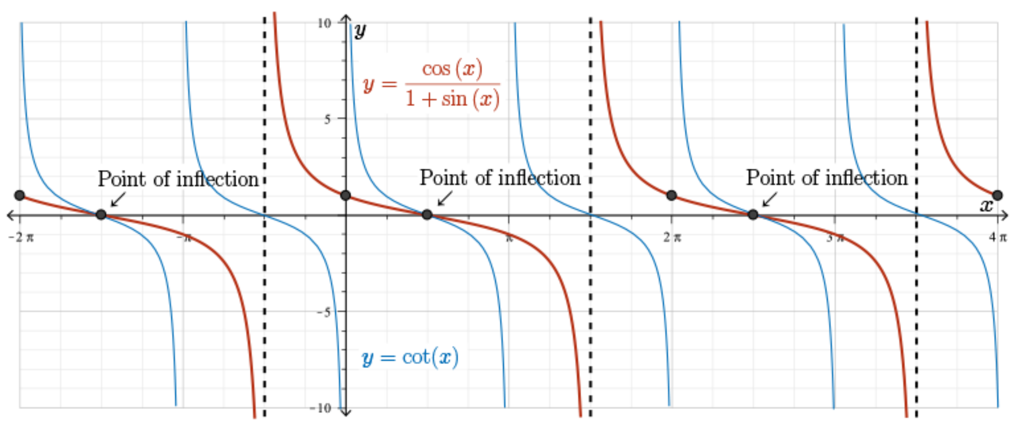


Examples

Example 2

Sketch the graph of the function $y = \dfrac{\cos(x)}{1+\sin(x)}$.

Solution



This graph will repeat with period 2π and so we get the above graph of $f(x)$.

Note the similarities and differences between this graph and the graph of $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$.

Examples

Example 3

Sketch the graph of the curve $y = \frac{e^x}{x}$.

Solution

Let $f(x) = \frac{e^x}{x}$.

1. The function $f(x)$ is a quotient of an exponential and a polynomial both having domain $\mathbb{R}$.
The function $f(x)$ is undefined only when the denominator is 0 and so the domain of $f(x)$ is all real numbers except $x = 0$.

2. Since $f(0)$ is undefined, $f(x)$ has no y-intercept.
That is, $f(x)$ does not cross the y-axis.

Examples

Example 3

Sketch the graph of the curve $y = \frac{e^x}{x}$.

Solution

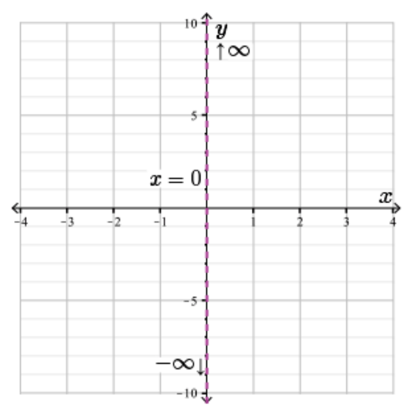
3. Since e^x and x are both continuous functions, $f(x)$ is continuous at every point in its domain.
There is only 1 discontinuity, occurring at $x = 0$.
Let's examine the behaviour of $f(x)$ as x approaches 0 from the left and from the right.

$$\text{As } x \to 0^+,\ f(x) = \frac{e^x}{x} \approx \frac{1}{0^+} \to +\infty.$$

$$\text{As } x \to 0^-,\ f(x) = \frac{e^x}{x} \approx \frac{1}{0^-} \to -\infty.$$

Therefore, $f(x)$ has a vertical asymptote at $x = 0$.

4. We have $f(x) \neq 0$ for all x in the domain of $f(x)$ and so $f(x)$ has no x-intercept.



Examples

Example 3

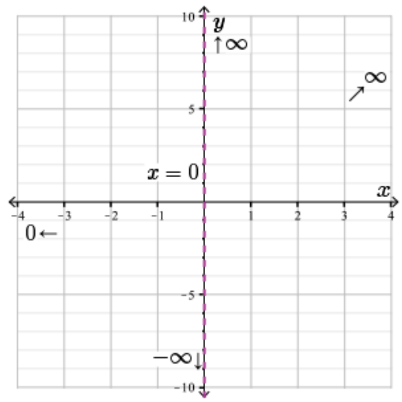
Sketch the graph of the curve $y = \frac{e^x}{x}$.

Solution

5-6. We need to examine the behaviour of the function as x approaches $\pm\infty$:

$$\begin{aligned} &\lim_{x\to\infty} \frac{e^x}{x} && \left(\frac{\infty}{\infty}\right) \\ &= \lim_{x\to\infty} \frac{\frac{d}{dx}\left(e^x\right)}{\frac{d}{dx}\left(x\right)} && \text{by l'Hospital's rule} \\ &= \lim_{x\to\infty} \frac{e^x}{1} \\ &\to \infty \end{aligned}$$

As $x \to -\infty$, $\frac{e^x}{x} \approx \frac{0^+}{-\infty} \to 0$



Since $f(x) < 0$ for $x < 0$, the function approaches the horizontal asymptote $y = 0$ from below.
Here, we have no oblique asymptote as the function increases exponentially as $x \to \infty$.

Examples

Example 3

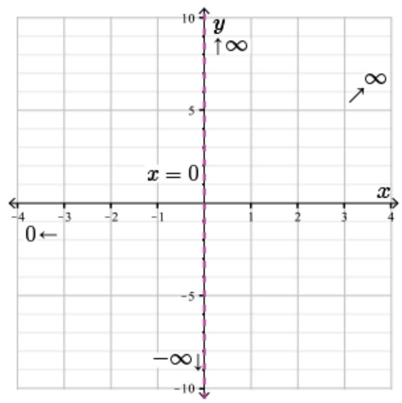
Sketch the graph of the curve $y = \frac{e^x}{x}$.

Solution

7. Next, we determine the critical points of $f(x)$ to locate the extreme values.
Using the quotient rule, we get

$$f'(x) = \frac{xe^x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$$

Since e^x and x^2 are positive for all $x \neq 0$, we have $f'(x) = 0$ if and only if $x = 1$.
Therefore, $f(x)$ has a local extreme at $x = 1$.
Since $f(x)$ (and $f'(x)$) is undefined at $x = 0$, we also have a critical point at $x = 0$.



Examples

Example 3

Sketch the graph of the curve $y = \frac{e^x}{x}$.

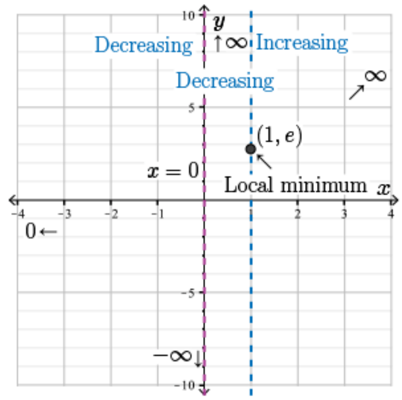
Solution

8. Let's examine the sign of the derivative on the intervals $x < 0$, $0 < x < 1$ and $x > 1$, determined by the critical points of $f(x)$.

Interval	$x < 0$	$0 < x < 1$	$x > 1$
$f'(x) = \frac{e^x(x-1)}{x^2}$	$\frac{(+)(-)}{(+)} < 0$	$\frac{(+)(-)}{(+)} < 0$	$\frac{(+)(+)}{(+)} > 0$
$f(x)$	Decreasing	Decreasing	Increasing

Therefore, there is a local *minimum* at $x = 1$ with value

$f(1) = \frac{e^1}{1} = e \approx 2.7$.



Examples

Example 3

Sketch the graph of the curve $y = \frac{e^x}{x}$.

Solution

9. Finally, we find all points at which $f(x)$ may change concavity. A concavity change may occur where $f''(x) = 0$ or where $f''(x)$ does not exist.

Since $f'(x) = \frac{e^x(x-1)}{x^2} = \frac{xe^x - e^x}{x^2}$ we find $f''(x)$ using the quotient rule as follows:

$$\begin{aligned} f''(x) &= \frac{x^2 \frac{d}{dx}\left[xe^x - e^x\right] - (xe^x - e^x)\frac{d}{dx}\left[x^2\right]}{(x^2)^2} \\ &= \frac{x^2\left[xe^x + e^x - e^x\right] - (xe^x - e^x)(2x)}{x^4} \\ &= \frac{x^3e^x - 2x^2e^x + 2xe^x}{x^4} \\ &= \frac{e^x(x^2 - 2x + 2)}{x^3} \end{aligned}$$

Since e^x and x^3 are both non-zero for all x in the domain of $f(x)$, we have $f''(x) = 0$ only if $x^2 - 2x + 2 = 0$.

Examples

Example 3

Sketch the graph of the curve $y = \frac{e^x}{x}$.

Solution

$$x^2 - 2x + 2 = 0$$

Using the quadratic formula, we solve this equation and find

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{-4}}{2}$$

and so this quadratic has no real roots. Therefore, we have $f''(x) \neq 0$ for all x in the domain of $f(x)$.

10. We can now determine the intervals on which the function is concave up and concave down.
Since $f(x)$ has no points of inflection, the only place where the concavity of $f(x)$ can change is at the discontinuity $x = 0$. Examining the sign of $f''(x)$ on the intervals $x < 0$ and $x > 0$ gives

Interval	$x < 0$	$x > 0$
$f''(x) = \frac{e^x(x^2 - 2x + 2)}{x^3}$	$f''(-1) = \frac{(+)(+)}{(-)} < 0$	$f''(1) = \frac{(+)(+)}{(+)} > 0$
$f(x)$	Concave down	Concave up

Examples

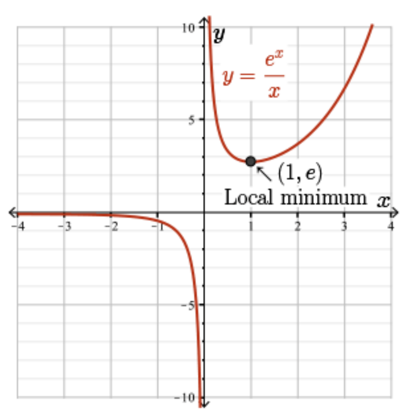
Example 3

Sketch the graph of the curve $y = \frac{e^x}{x}$.

Solution

Interval	$x < 0$	$x > 0$
$f(x)$	Concave down	Concave up

11. Using all of the information that we have gathered, we make the following sketch.



Examples

Challenge Question

Sketch the graph of the curve $y = \dfrac{\ln(x) + x}{e^x}$.

Solution

Let $f(x) = \dfrac{\ln(x) + x}{e^x}$.

1. Since the domain of $\ln(x)$ is $x > 0$, the domain of $\ln(x) + x$ is $x > 0$.
Since the domain of e^x is $\mathbb{R}$ and $e^x \neq 0$, the domain of $f(x)$ is also $x > 0$.

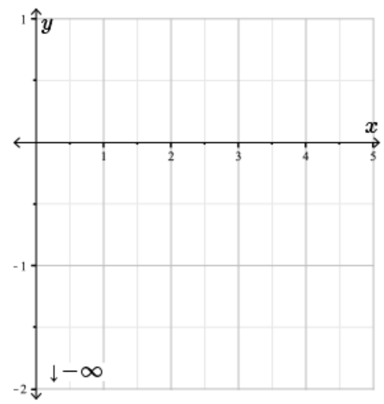
2. Since $f(0)$ is not defined, the function $f(x)$ has no y-intercept.

3. Since $\ln(x)$, x and e^x are continuous for all $x > 0$ and $e^x \neq 0$, the function $f(x)$ is continuous at every point in its domain.

We need to examine the behaviour of $f(x)$ as x approaches the endpoint $x = 0$ from the right.
As $x \to 0^+$, we have $\ln(x) + x \to -\infty$ and $e^x \to 1$ and so

$$f(x) = \frac{\ln(x) + x}{e^x} \to -\infty \text{ as } x \to 0^+$$

Therefore, $f(x)$ has a vertical asymptote at $x = 0$.



Examples

Challenge Question

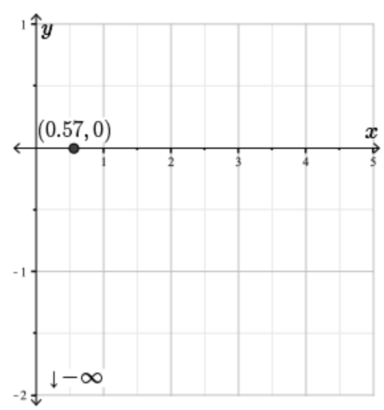
Sketch the graph of the curve $y = \dfrac{\ln(x) + x}{e^x}$.

Solution

4. Since $e^x > 0$ for all x, we have $f(x) = \dfrac{\ln(x) + x}{e^x} = 0$ exactly when $\ln(x) + x = 0$.
How many roots does this equation have and how do we locate the roots?
Note that the roots of this equation are the points x that satisfy $\ln(x) = -x$.
From a sketch of the two functions in question, we conclude that there is exactly 1 root, and that the root lies in the interval $[0, 1]$.
We will use Newton's method to approximate the root to 2 decimal places, which should be sufficient for our sketch of $f(x)$.
Using $g(x) = \ln(x) + x$ and $x_1 = 1$, try Newton's method now.

Newton's method with $g(x) = \ln(x) + x$, $g'(x) = \frac{1}{x} + 1$ and $x_1 = 1$ produces the sequence

$$x_1 = 1,\ x_2 = 0.500,\ x_3 \approx 0.564,\ x_4 \approx 0.567,\ x_5 \approx 0.567, \ldots$$



We conclude that the root of $\ln(x) + x = 0$, and hence the root of $f(x) = 0$, occurs at $x \approx 0.57$.

Examples

Challenge Question

Sketch the graph of the curve $y = \dfrac{\ln(x) + x}{e^x}$.

Solution

5-6. Next, we need to determine the behaviour of $f(x)$ as x approaches ∞.

Since $f(x)$ is only defined for $x > 0$, we do not need to consider the limit as x approaches $-\infty$.

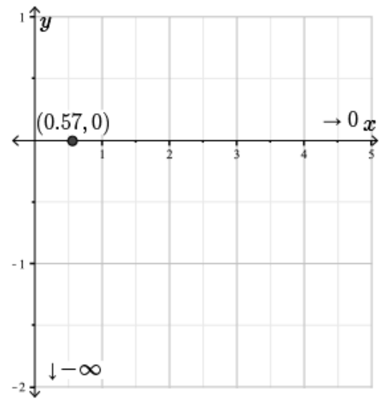
We have

$$\begin{aligned} &\lim_{x\to\infty} \frac{\ln(x) + x}{e^x} && \left(\frac{\infty}{\infty}\right) \\ &= \lim_{x\to\infty} \frac{\frac{1}{x} + 1}{e^x} && \left(\frac{1}{\infty}\right) \\ &= 0 \end{aligned}$$

using l'Hospital's rule.

Since $\ln(x) + x > 0$ for all $x \geq 1$ and $e^x > 0$ for all x, we have $f(x) > 0$ for all $x \geq 1$.

Therefore, the function approaches the horizontal asymptote $y = 0$ from above.



Examples

Challenge Question

Sketch the graph of the curve $y = \dfrac{\ln(x) + x}{e^x}$.

Solution

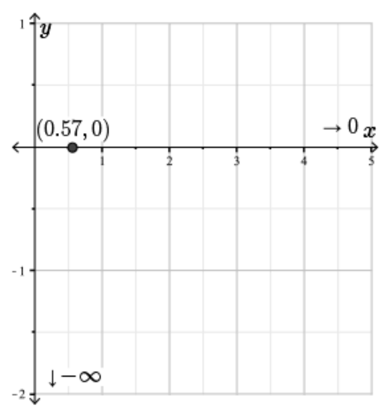
7. Let's find the critical points of $f(x)$.

Using the quotient rule, we find the derivative of $f(x)$:

$$\begin{aligned} f'(x) &= \frac{e^x\left(\frac{1}{x} + 1\right) - (\ln(x) + x)e^x}{(e^x)^2} \\ &= \frac{e^x\left(\frac{1}{x} - \ln(x) - x + 1\right)}{(e^x)^2} \\ &= \frac{\frac{1}{x} - \ln(x) - x + 1}{e^x} \end{aligned}$$

We have $f'(x) = 0$ exactly when $\dfrac{1}{x} - \ln(x) - x + 1 = 0$ or, equivalently, when

$$\ln(x) = \frac{1}{x} - x + 1 = \frac{1 - x^2 + x}{x}$$



Examples

Challenge Question

Sketch the graph of the curve $y = \dfrac{\ln(x) + x}{e^x}$.

Solution

$$\ln(x) = \frac{1 - x^2 + x}{x}$$

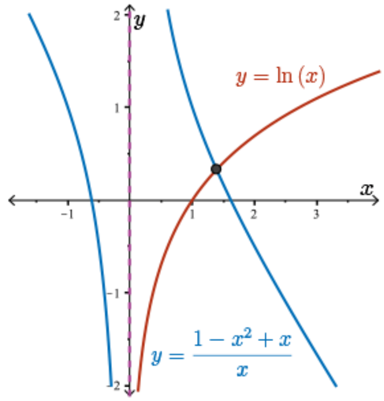
Sketching the function $\ln(x)$ and the rational function $\dfrac{1 - x^2 + x}{x}$, we see that the two functions intersect at exactly 1 point, and hence there is 1 real root of the equation $f'(x) = 0$.

The root is in the interval $[1, 2]$, and we use Newton's method with $g(x) = \dfrac{1}{x} - \ln(x) - x + 1$ to find the root correct to 2 decimal places: $x \approx 1.39$.

Try this on your own!

Therefore, the root of $f'(x) = 0$ is $x \approx 1.39$.

Since $f'(x)$ is defined for all $x > 0$, the only critical point of $f(x)$ is $x \approx 1.39$, which is a local extreme.



Examples

Challenge Question

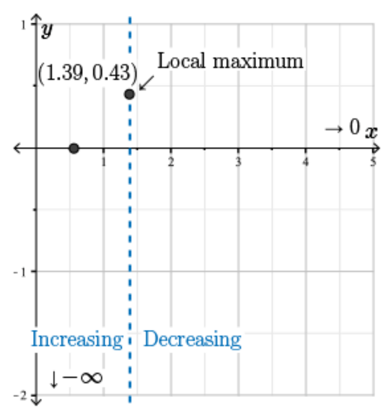
Sketch the graph of the curve $y = \dfrac{\ln(x) + x}{e^x}$.

Solution

8. Next we will examine the sign of the derivative on the intervals $0 < x < 1.39$ and $x > 1.39$.

Interval	$0 < x < 1.39$	$x > 1.39$
$f'(x)$	$f'(1) > 0$	$f'(2) < 0$
$f(x)$	Increasing	Decreasing

Therefore, the point $x \approx 1.39$ is a local maximum, and the maximum value is $f(1.39) \approx 0.43$.



Examples

Challenge Question

Sketch the graph of the curve $y = \dfrac{\ln(x) + x}{e^x}$.

Solution

9. Finally, we find all points at which $f(x)$ may change concavity.

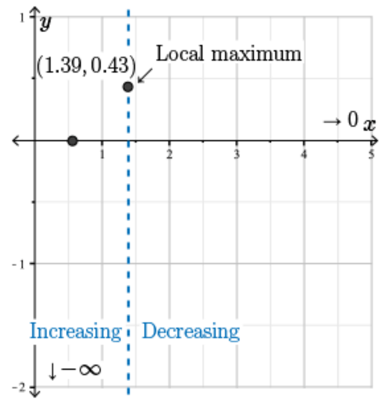
Since $f'(x) = \dfrac{\frac{1}{x} - \ln(x) - x + 1}{e^x}$, using the quotient rule, we have

$$f''(x) = \frac{e^x\left(-\frac{1}{x^2} - \frac{1}{x} - 1\right) - \left(\frac{1}{x} - \ln(x) - x + 1\right)e^x}{(e^x)^2}$$

$$= \frac{e^x\left(-\frac{1}{x^2} - \frac{1}{x} - 1 - \frac{1}{x} + \ln(x) + x - 1\right)}{(e^x)^2}$$

$$= \frac{-\frac{1}{x^2} - \frac{2}{x} + \ln(x) + x - 2}{e^x}$$

The function $f''(x)$ is defined for all $x > 0$ and $f''(x) = 0$ if and only if $-\frac{1}{x^2} - \frac{2}{x} + \ln(x) + x - 2 = 0$.



As in Step 7, we can use a sketch of $\ln(x)$ and an appropriate rational function to determine that this equation has exactly 1 real root, and we use Newton's method to approximate the root to 2 decimal places.
Doing so, we find that $f''(x) = 0$ when $x \approx 2.26$.

Examples

Challenge Question

Sketch the graph of the curve $y = \dfrac{\ln(x) + x}{e^x}$.

Solution

10. Examining the sign of the second derivative on the intervals $0 < x < 2.26$ and $x > 2.26$, we get the following:

Interval	$0 < x < 2.26$	$x > 2.26$
$f''(x)$	$f''(2) < 0$	$f''(3) > 0$
$f(x)$	Concave down	Concave up

11. Using all of the information that we have gathered, we make the following sketch.