Volumes

Introduction

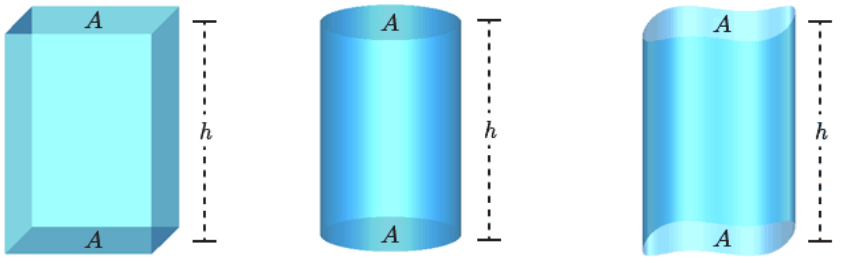
Definite integrals were introduced as a way to calculate areas of regions in the plane.
It turns out that they can also be used to calculate volumes of solids!

In this module, we will revisit the theory of integration and discuss how this can be used to calculate certain volumes.

Volumes of Simple Solids

Consider the following three solids. We are interested in what these solids have in common.

Rectangular box: $A = l \times w$ Right circular cylinder: $A = \pi r^2$ General base with area A



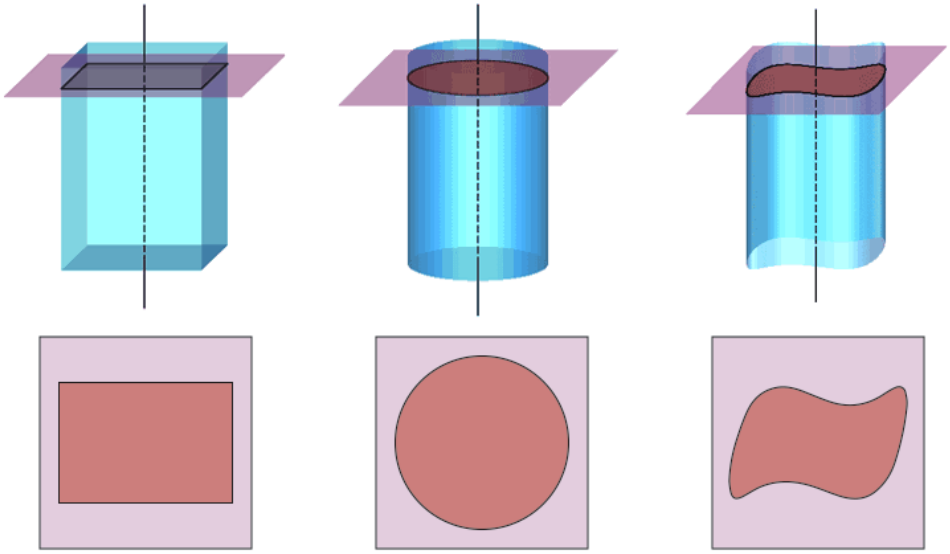
In each solid

- the top and bottom surfaces are congruent shapes that lie in parallel planes,
- the sides of the solid meet the top and bottom surfaces at right angles, and
- if you intersect the solid with any plane parallel to the plane containing the base, then the region of intersection is, again, congruent to the top and bottom surfaces.

Volumes of Simple Solids

Cross Sections

Note: A two-dimensional region obtained by intersecting the solid with a plane is called a **cross section** of the solid.



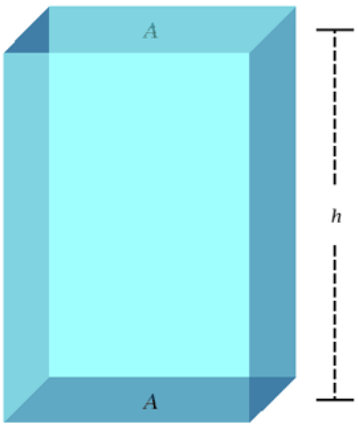
Volumes of Simple Solids

The volume of these solids can be calculated using the same formula:

$$\text{Volume of the solid} = \text{cross-sectional area} \times \text{height} = Ah$$

For example (familiar special cases of this general formula),

- a rectangular box has a volume of $V = Ah = lwh$, and



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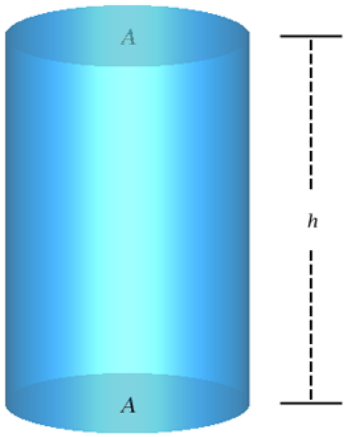

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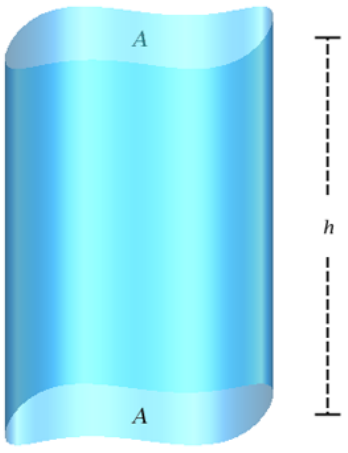
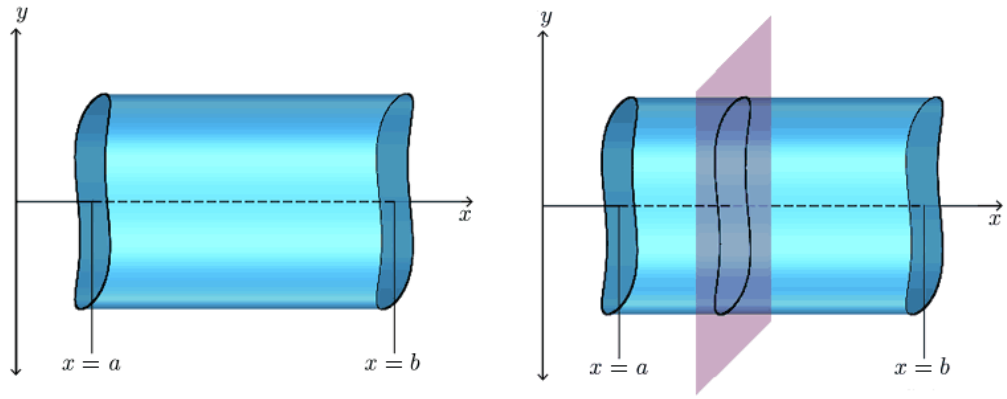
- a right circular cylinder has a volume of $V = Ah = \pi r^2 h$.

Note: Any solid with the properties that we described is called a **right cylinder**. Keep in mind that the bases of a right cylinder do not have to be circular.

Question: How do we compute the volume of solids that are not right cylinders?



Volumes of Simple Solids

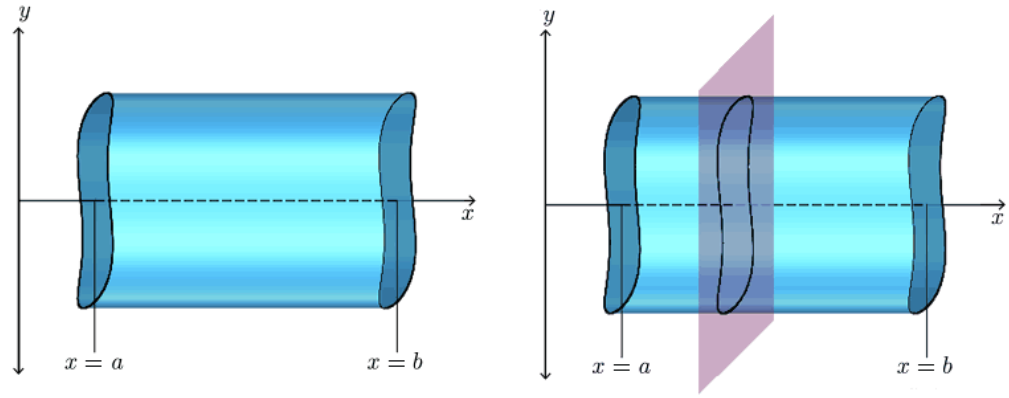


Let's rotate the more general right cylinder and introduce some axes. Here, we see our general right cylinder which is sketched so that it is contained between the two parallel planes $x = a$ and $x = b$ (that is, the top and bottom surfaces lie in these vertical planes).

Then, for all x in the interval $[a, b]$, we define the cross-sectional area function, $A(x)$, as follows:

$A(x)$ = cross-sectional area obtained when the solid is intersected with a plane parallel to the plane $x = a$ that passes through the point x.

Volumes of Simple Solids



In this example (for the right cylinder), the function $A(x)$ is *constant*.
In particular, $A(x) = area\ of\ base = A$ for all $a \leq x \leq b$.

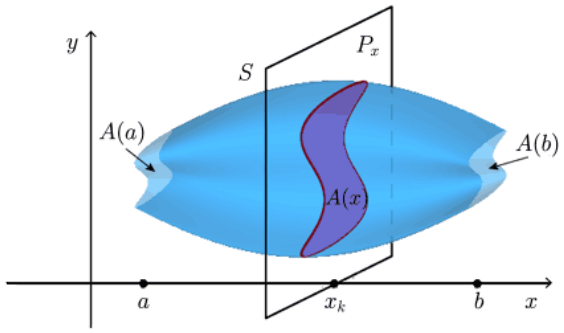
Let's reformulate the volume in terms of this area function and the parameters a and b:

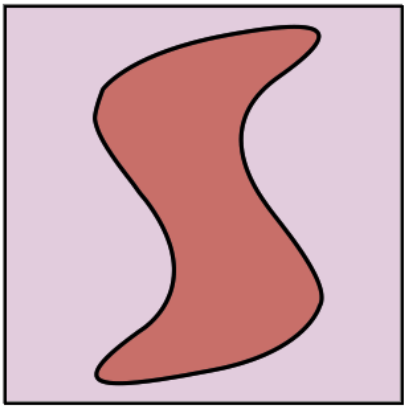
$$V = Ah = A(b - a) = A\int_a^b 1\,dx = \int_a^b A\,dx = \int_a^b A(x)\,dx$$

Volumes of Simple Solids

$$V = \int_a^b A(x)\,dx$$

It turns out that this formula will compute the volume of a solid even if the cross-sectional areas are not constant.





Volumes of Simple Solids

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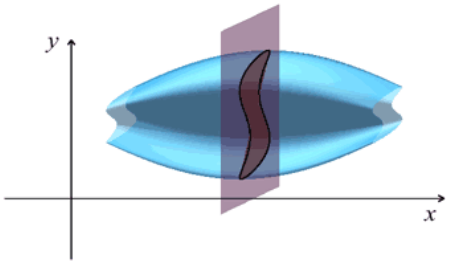
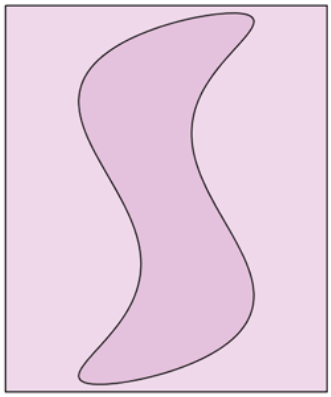




Volumes of Simple Solids

Definition

Volume of a Solid

Suppose that a solid S is contained between the parallel planes $x = a$ and $x = b$. Then, the volume of S is given by

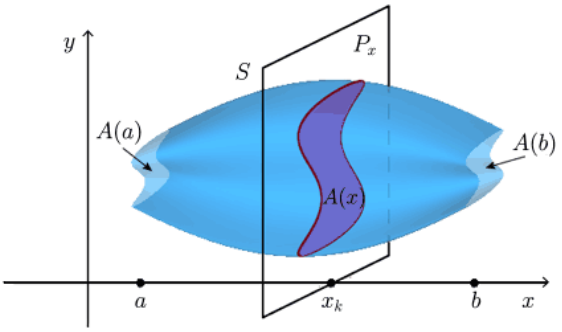
$$V = \int_a^b A(x)\,dx$$

where $A(x)$, for $a \leq x \leq b$, is the cross-sectional area obtained when the plane P_x (parallel to $x = a$ and passing through x) is intersected with S.

The formal proof of this formula requires a discussion of Riemann sums. We will provide a sketch of this proof.

Suppose that S is contained between $x = a$ and $x = b$ and let $A(x)$ be the cross-sectional area function defined for $a \leq x \leq b$.

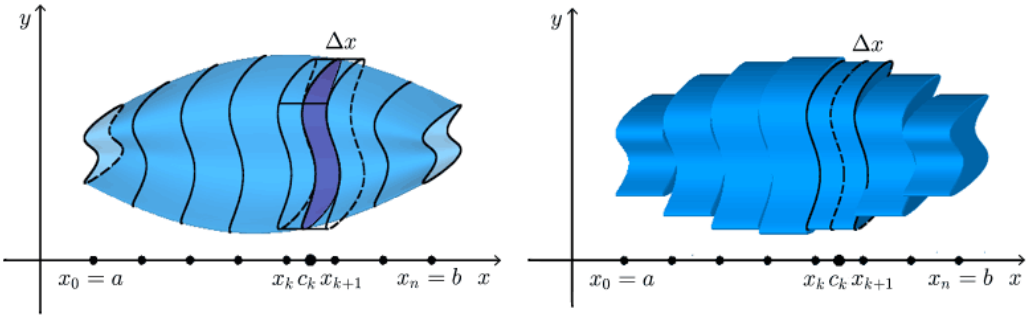
Recall: To estimate the area of an irregular region in the plane, we divided the region into vertical strips of uniform width and approximated each vertical strip using a shape with a known area (rectangle).



Volumes of Simple Solids

How do we apply this reasoning to estimate a volume?

Here, we will divide the solid into vertical slices and approximate each slice using a solid with a known volume (right cylinder).



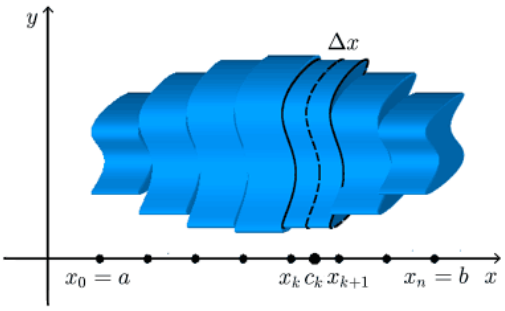
We can formalize this as follows: Partition the interval $[a, b]$ into n equal subintervals of length Δx. This divides the solid into n slices, and each slice will be approximated using a right cylinder.

Volumes of Simple Solids

1. The heights of all right cylinders will be constant and equal to Δx (the length of the subintervals).

2. The cross-sectional areas of the right cylinders will vary. The cross-sectional area of the k^{th} cylinder will be determined by choosing a sample point, c_k, in the k^{th} subinterval. The cross-sectional area of the k^{th} approximating cylinder will be $A(c_k)$.

The sum of the volumes of the n cylinders gives us an approximation for the volume of the solid S:



$$V \approx \sum_{k=1}^{n} \underbrace{A(c_k)\,\Delta x}_{V(k^{th}\ cylinder)}$$

Observe that this sum is a Riemann sum of the function $A(x)$ over the interval $[a, b]$.
As we let n get larger and larger, we expect to get better and better approximations of the volume of the solid.

In the limit, as $n \to \infty$, these sums approach the exact value of the volume and we get the desired definite integral:

$$V = \int_a^b A(x)\,dx$$

Examples

Example 1

Show that the volume of a sphere of radius 4 is given by $V = \frac{4}{3}\pi(4)^3$.

Solution

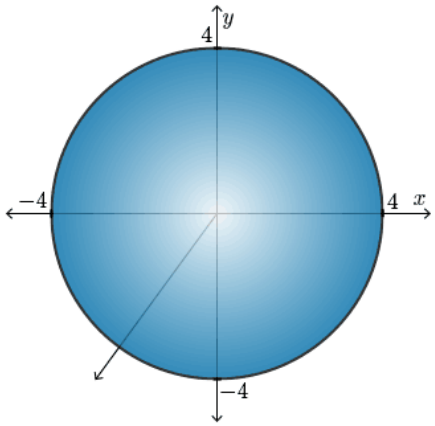
We would like to use the formula we derived for computing volumes:

$$V = \int_a^b A(x)\,dx$$

We need to determine a cross-sectional area function, $A(x)$, for the sphere and identify the endpoints, a and b.

Let's centre the sphere at the origin. Since the sphere has radius 4, we see that the sphere is contained between the vertical planes $x = -4$ and $x = 4$.

Therefore, $a = -4$ and $b = 4$.



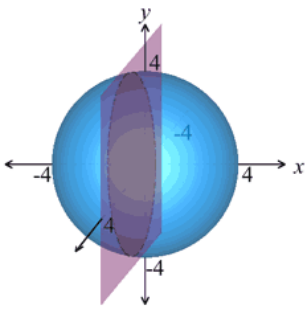
Examples

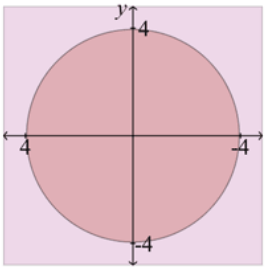
Example 1

Show that the volume of a sphere of radius 4 is given by $V = \frac{4}{3}\pi(4)^3$.

Solution

Now, we consider all vertical planes (parallel to $x = a$) and their cross sections of the sphere.





We see that each cross section is a disk of a particular radius.
When $x = -4$ the cross section is a point (which we can think of as a disk of radius 0).
Then, the radius increases from 0 at $x = -4$ to 4 at $x = 0$, and then decreases back to 0 at $x = 4$.

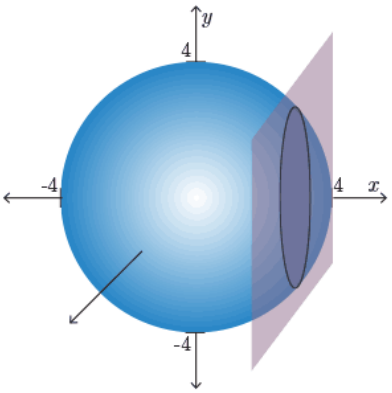
Examples

Example 1

Show that the volume of a sphere of radius 4 is given by $V = \frac{4}{3}\pi(4)^3$.

Solution

What is an explicit formula for the radius for each x in $[-4, 4]$?



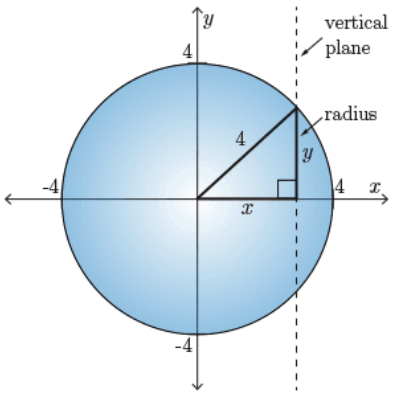
Examples

Example 1

Show that the volume of a sphere of radius 4 is given by $V = \frac{4}{3}\pi(4)^3$.

Solution

What is an explicit formula for the radius for each x in $[-4, 4]$?
Let's fix an x-value in the interval $(-4, 4)$. Consider the following right-angled triangle in the xy-plane.



We have that,

- the distance from the origin to the surface of the sphere is equal to 4, the radius of the sphere,
- the horizontal distance from the origin to the cross section is the value x, and
- the radius of the cross-sectional circle is the length of the third side of this triangle. This quantity is labelled y.

By the Pythagorean theorem, we have

$$y = \sqrt{4^2 - x^2} = \sqrt{16 - x^2}$$

This is an explicit formula for the radius of each cross-sectional disk.

Examples

Example 1

Show that the volume of a sphere of radius 4 is given by $V = \frac{4}{3}\pi(4)^3$.

Solution

A formula for the area of each disk is given by the following:

$$A(x) = \pi y^2 = \pi\left(\sqrt{16 - x^2}\right)^2 = \pi(16 - x^2), \quad -4 \leq x \leq 4$$

where y is the cross-sectional radius.
Using the formula for volume, we have the following:

$$V = \int_a^b A(x)\,dx = \int_{-4}^{4} \pi(16 - x^2)\,dx$$

Evaluating this integral gives the volume of the sphere:

$$V = \pi\left[16x - \frac{x^3}{3}\right]_{-4}^{4} = \pi\left(128 - \frac{128}{3}\right) = \pi\left(\frac{256}{3}\right) = \pi\,\frac{(4)^4}{3} = \frac{4}{3}\pi(4)^3$$

Observe that we can replace the radius 4 with the more general radius r and we get that the volume of a sphere of radius r is given by $V = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = \frac{4}{3}\pi r^3$. See if you can derive this more general formula using the above method.

Volumes of Revolution

For the remainder of this module, we will focus on computing volumes of less familiar solids.
In particular, we will calculate volumes of solids obtained by rotating two-dimensional regions about either the x-axis or the y-axis.

Volumes of Revolution

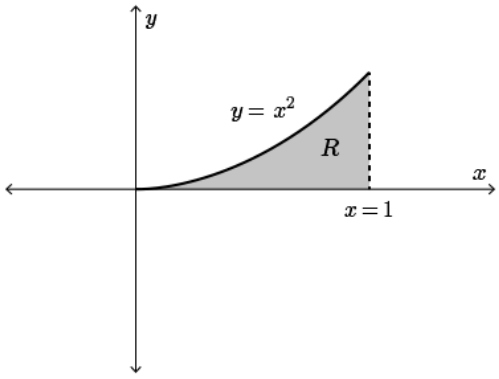
Example 2

Let R be the two-dimensional region bounded by the curve $y = x^2$ and the x-axis, from $x = 0$ to $x = 1$. Find the volume of the solid S obtained by rotating the region R about the x-axis.

Solution

First, we sketch the parabola $y = x^2$ from $x = 0$ to $x = 1$ and identify the shaded region R.

What happens if we rotate this region about the x-axis? This region will "sweep out" a three-dimensional solid.



Volumes of Revolution

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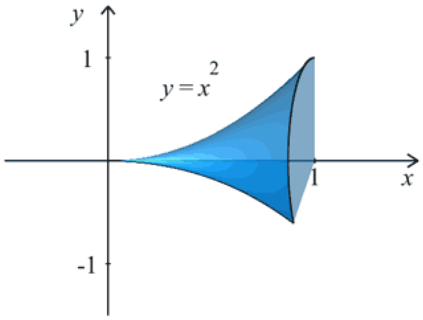


What happens if we rotate this region about the x-axis? This region will "sweep out" a three-dimensional solid.

Observe that rotating this region forms a solid that looks like a funnel.

What is the volume of this solid S?

We will use the same formula again, which means finding a formula for the cross-sectional area and the endpoints a and b.

We see that the solid S is contained between the vertical planes $x = 0$ and $x = 1$.

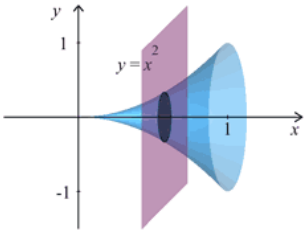
Volumes of Revolution

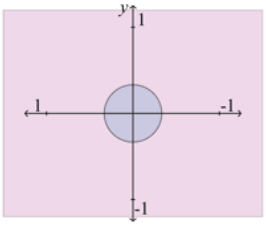
Example 2

Let R be the two-dimensional region bounded by the curve $y = x^2$ and the x-axis, from $x = 0$ to $x = 1$. Find the volume of the solid S obtained by rotating the region R about the x-axis.

Solution

If we fix a point x in $[a, b]$, and intersect S with the vertical plane P_x, we get the following cross section of S:





Again, the cross sections are all *disks*. This time, the radii vary from 0 (at $x = 0$) to 1 (at $x = 1$).

From the picture, we see that the radius of the cross section at some x-value is given by the height of the function $y = x^2$ at this x-value. In other words,

$$\text{cross-sectional radius} = y = x^2$$

Volumes of Revolution

Example 2

Let R be the two-dimensional region bounded by the curve $y = x^2$ and the x-axis, from $x = 0$ to $x = 1$. Find the volume of the solid S obtained by rotating the region R about the x-axis.

Solution

Therefore, we have

$$A(x) = \pi\left(x^2\right)^2 = \pi x^4,\ 0 \leq x \leq 1$$

where x^2 is the cross-sectional radius.
Using the formula for calculating volumes, we have

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi x^4\,dx = \pi\left[\frac{x^5}{5}\right]_0^1 = \frac{\pi}{5}$$

Volumes of Revolution

Example 3

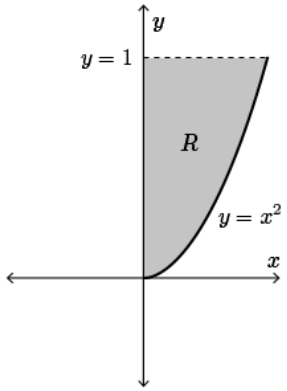
Let R be the region bounded by the curve $y = x^2$ and the y-axis from $y = 0$ to $y = 1$. Find the volume of the solid S obtained by rotating the region R about the y-axis.

Solution

In this problem, the description of the region is given in terms of y, not x, so we will attempt to calculate the volume using y as the variable of integration.

The shaded region R is bounded below by $y = x^2$, above by $y = 1$, and on the left by the y-axis.

What happens when we rotate this region about the y-axis?



Volumes of Revolution

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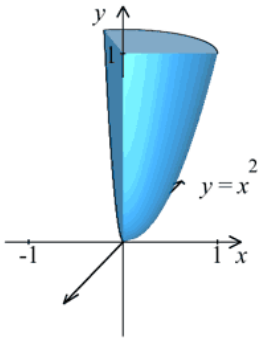
The shaded region R is bounded below by $y = x^2$, above by $y = 1$, and on the left by the y-axis.

What happens when we rotate this region about the y-axis?

Again, we "sweep out" a three-dimensional solid.

This time we form a solid that looks like a bowl.



Since we are rotating the region about the y-axis, it makes more sense to consider horizontal cross sections, instead of vertical cross sections. (What would a formula for $A(x)$ be?)

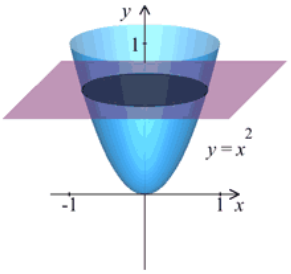
Volumes of Revolution

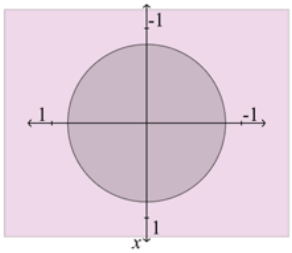
Example 3

Let R be the region bounded by the curve $y = x^2$ and the y-axis from $y = 0$ to $y = 1$. Find the volume of the solid S obtained by rotating the region R about the y-axis.

Solution

Following a similar method (with y in place of x), we intersect the solid S with a *horizontal* plane P_y, passing through the point y, and find a formula for the *horizontal* cross-sectional area, $A(y)$.





As we can see, each horizontal cross section is, again, a disk.

Volumes of Revolution

Example 3

Let R be the region bounded by the curve $y = x^2$ and the y-axis from $y = 0$ to $y = 1$. Find the volume of the solid S obtained by rotating the region R about the y-axis.

Solution

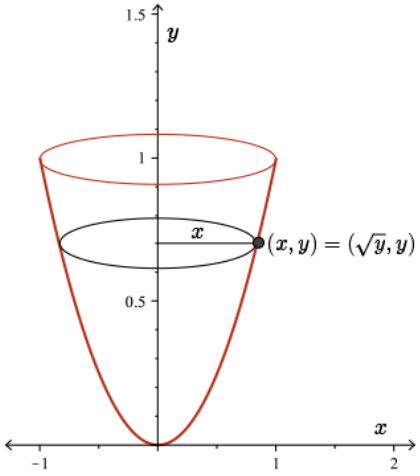
Fix y in $(0, 1)$.

Each horizontal cross section is a disk with radius increasing from 0 (at $y = 0$) to 1 (at $y = 1$).

What is the radius of this particular cross section?

It is equal to the horizontal distance from the y-axis to the curve $y = x^2$.

If we consider the point (x, y) on the curve, then the radius is the value of x.



Volumes of Revolution

Example 3

Let R be the region bounded by the curve $y = x^2$ and the y-axis from $y = 0$ to $y = 1$. Find the volume of the solid S obtained by rotating the region R about the y-axis.

Solution

Since we are looking for an area function in terms of y, we solve for x in the equation of the curve and obtain

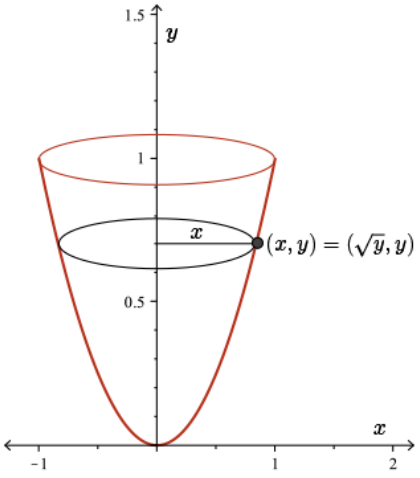
$$\text{cross-sectional radius} = x = \sqrt{y}$$

Therefore, our cross-sectional area function is

$$A(y) = \pi\left(\sqrt{y}\right)^2 = \pi y$$

Using the analogous formula for calculating volumes, in terms of the variable y, we get

$$V = \int_0^1 A(y)\,dy = \int_0^1 \pi\, y\, dy = \pi\left[\frac{y^2}{2}\right]_0^1 = \frac{\pi}{2}$$

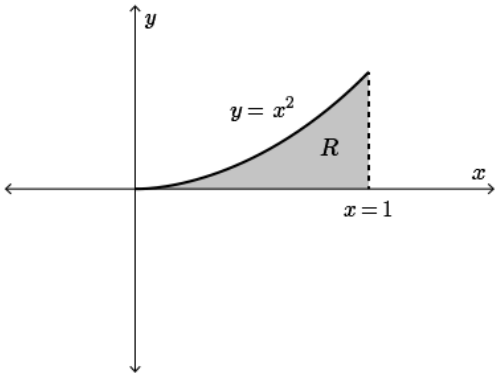


Summary

Now, let's summarize what we have discussed in terms of volumes of revolution.

Volumes of Revolution

Let $f(x)$ be continuous and non-negative over the interval $a \leq x \leq b$, and let R be the region bounded by the curve $y = f(x)$ and the x-axis over $[a, b]$.



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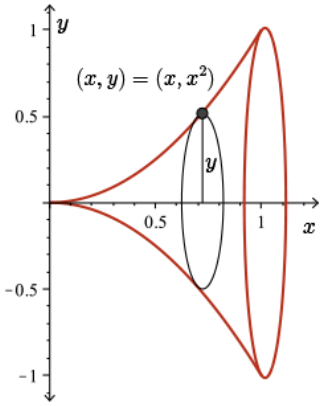
Let $f(x)$ be continuous and non-negative over the interval $a \leq x \leq b$, and let R be the region bounded by the curve $y = f(x)$ and the x-axis over $[a, b]$.

The volume of the solid obtained by rotating the region R about the x-axis is given by

$$V = \int_a^b \pi \left(\text{radius of circular cross section}\right)^2 dx$$
$$= \int_a^b \pi (f(x))^2 \, dx$$

$$V = \int_0^1 \pi (f(x))^2 \, dx = \int_0^1 \pi (x^2)^2 \, dx$$



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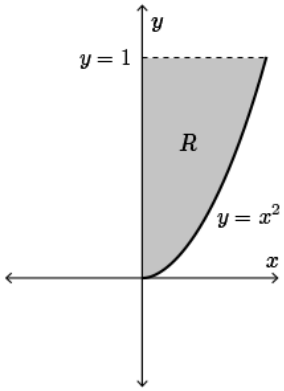
Now, let's summarize what we have discussed in terms of volumes of revolution.

Volumes of Revolution

Let $g(y)$ be continuous and non-negative over the interval $c \leq y \leq d$, and let R be the region bounded by the curve $x = g(y)$ and the y-axis over $[c, d]$.



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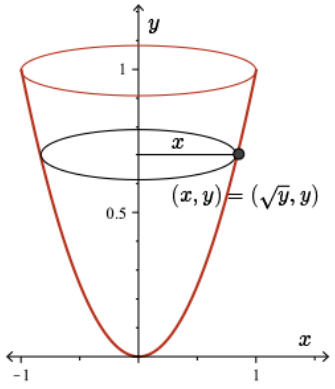


Challenge Question

Let R be the region enclosed by the curve $y = \sqrt{x}$ and the horizontal line $y = 1$, from $x = 0$ to $x = 1$. Find the volume of the solid obtained when the region R is rotated about the x-axis.

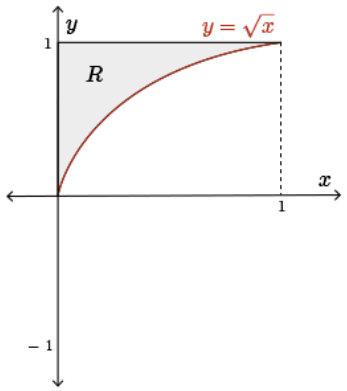
Solution

As usual, we start by sketching the region. We draw the curve $y = \sqrt{x}$ and the line $y = 1$.

We are interested in the region below the line and above the curve from $x = 0$ to $x = 1$.

Note that these curves intersect at the point $(1, 1)$.

What solid is formed when we rotate this region about the x-axis?



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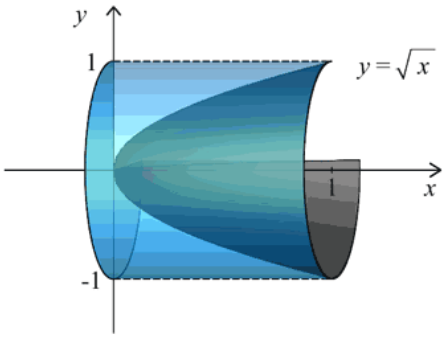
The line $y = 1$ "sweeps" out the outer surface of a cylinder, while the curve $y = \sqrt{x}$ sweeps out an inner "parabolic" bowl.

The resulting solid is essentially a right circular cylinder, of radius 1 and height 1, with the middle scooped out.



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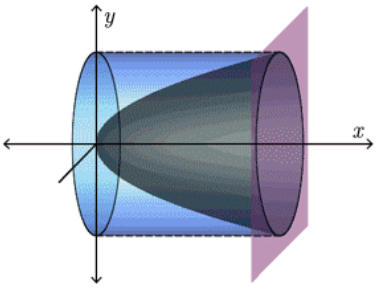
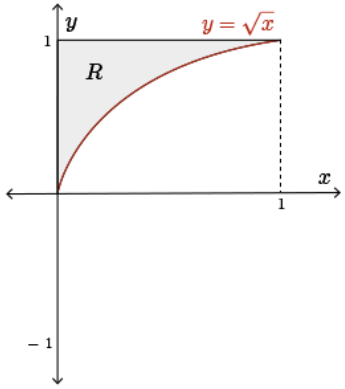
Solution

The solid is contained between the vertical planes $x = 0$ and $x = 1$.

What about the cross-sectional area function?

Fix an x-value, $0 \leq x \leq 1$, and intersect the solid with a vertical plane at that x value.





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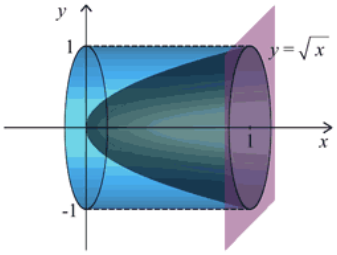

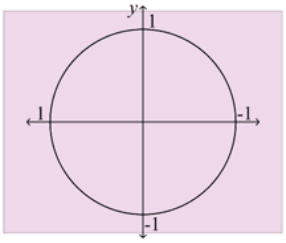

Each cross section is a disk with an inner portion removed.

The inner portion is also a disk.

The area of these cross sections can be calculated as the difference of two areas: the area of the outer disk minus the area of the inner disk.

Challenge Question

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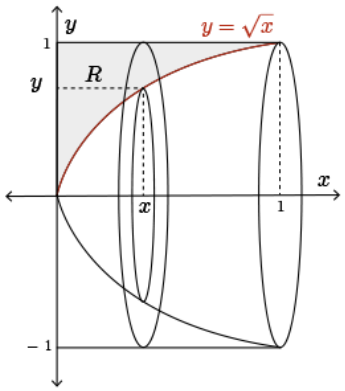
Solution

The radius of the outer disk is always equal to 1.

The radius of the inner disk varies but is always equal to the height of the function $y = \sqrt{x}$.

We have the following formula for the cross-sectional area:

$$\begin{aligned} A(x) &= \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 \\ &= \pi(1)^2 - \pi(\sqrt{x})^2 = \pi(1 - x) \end{aligned}$$

It follows that the volume of the solid is

$$\begin{aligned} V &= \int_0^1 A(x)\, dx \\ &= \int_0^1 \pi(1 - x)\, dx \\ &= \pi\left[x - \frac{x^2}{2}\right]_0^1 \\ &= \pi\left(1 - \frac{1}{2}\right) = \frac{\pi}{2} \end{aligned}$$