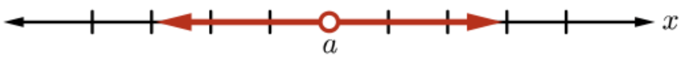
Evaluating Limits Graphically

Criteria for a Limit to Exist

The term **limit** asks us to find a value that is approached by $f(x)$ as x approaches a, but does not equal a.
This value is written as

$$\lim_{x \to a} f(x)$$

To consider this limit, f must be defined at all points in some interval around $x = a$, although not necessarily at $x = a$.



A limit may not always exist at the point $x = a$.
For $\lim_{x \to a} f(x)$ to exist, $f(x)$ must approach the same value as x approaches a from the left, denoted $\lim_{x \to a^-} f(x)$, and as x approaches a from the right, denoted $\lim_{x \to a^+} f(x)$.

Criteria for a Limit to Exist

More precisely, for $\lim_{x \to a} f(x)$ to exist, the following three conditions must be met:

$\lim_{x \to a^-} f(x)$ must exist,

$\lim_{x \to a^+} f(x)$ must exist, and

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

To begin, we shall explore this concept graphically by examining the behaviour of the graph of $f(x)$ near $x = a$ for a variety of functions.

Examples

Example 1

Left-Hand Limit

The value that $f(x)$ approaches as x moves along the graph from the left side is the **left-hand limit**.

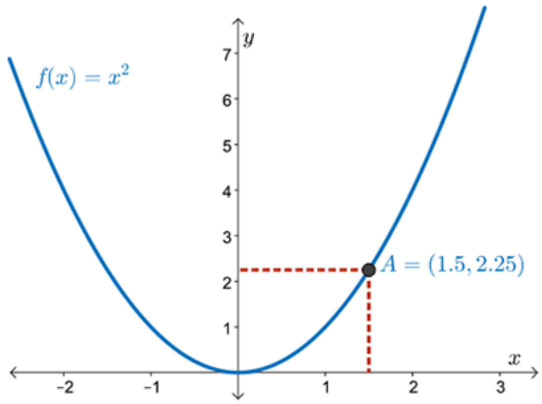
Consider the graph of $f(x) = x^2$.

If we want to find the left-hand limit of $f(x)$ as x approaches 2, we begin at a point on the parabola just left of $x = 2$.

Let's begin at $x = 1.5$.

$$\lim_{x \to 2^-} f(x) = \ ?$$

$\uparrow$ as x appproaches 2 from the left



Examples

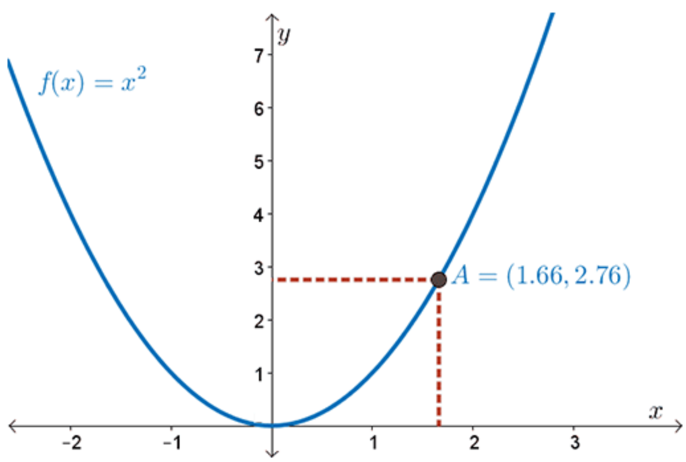
Example 1

Left-Hand Limit

As x moves along the graph from the left side, we see that $f(x)$ approaches the value 4.

$$\lim_{x \to 2^-} f(x) = 4$$

$\uparrow$ as x appproaches 2 from the left



Examples

Example 1

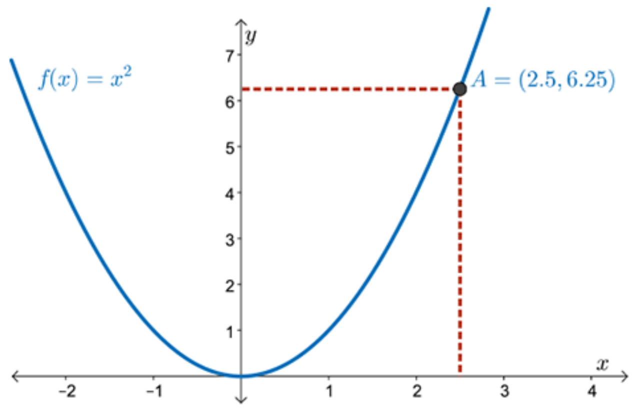
Right-Hand Limit

The value that $f(x)$ approaches as x moves along the graph from the right side is the **right-hand limit**.
To find the right-hand limit of $f(x)$ as x approaches 2, we begin at a point on the parabola just right of $x = 2$.
Let's begin at $x = 2.5$.

$$\lim_{x \to 2^+} f(x) = \ ?$$

$\uparrow$

as x appproaches 2 from the right



Examples

Example 1

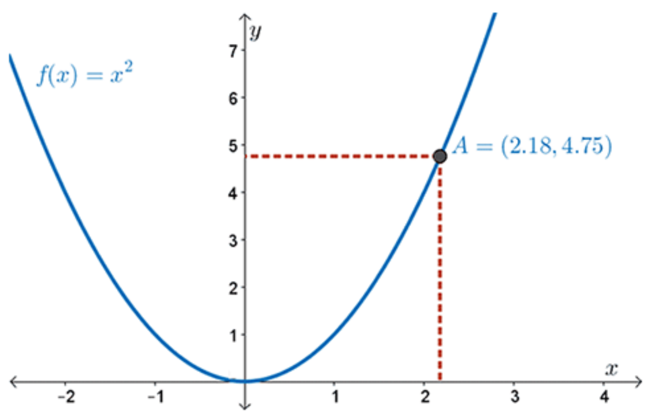
Right-Hand Limit

As x moves along the graph from the right side, we see that $f(x)$ approaches the value 4.

$$\lim_{x \to 2^+} f(x) = 4$$

$\uparrow$

as x appproaches 2 from the right



Examples

Example 1

Evaluating the Limit From Both Sides

We have seen that the limit exists as we approach the value $x = 2$ for $f(x)$ from the left side and the right side. Also, as x moves along the graph from the left and right sides of 2, we see that $f(x)$ approaches the value 4 from both directions.

Therefore, the limit of $f(x)$ from the left side is equal to the limit of $f(x)$ from the right side.

In summary, the following three conditions have been met:

$$\lim_{x \to 2^-} f(x) \text{ exists and equals } 4$$

$$\lim_{x \to 2^+} f(x) \text{ exists and equals } 4$$

$$\therefore \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 4$$

Thus, $\lim_{x \to 2} f(x)$ exists and equals 4.

Examples

Example 1

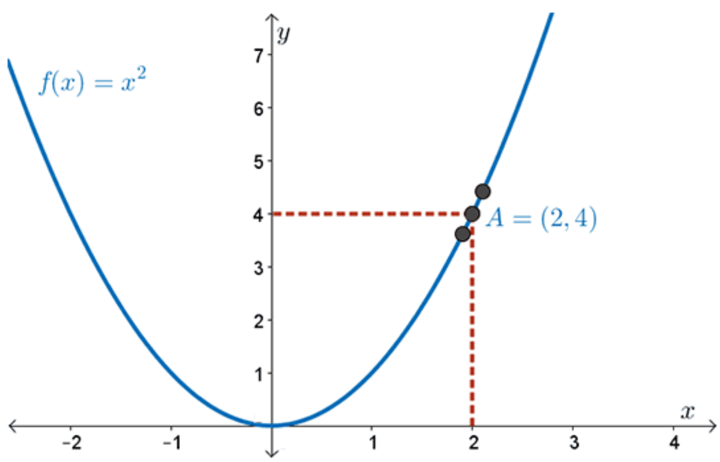
Evaluating the Limit From Both Sides

Since $f(x)$ approaches the value 4 from both directions,

$$\lim_{x \to 2} f(x) = 4$$

$\uparrow$

as x appproaches 2

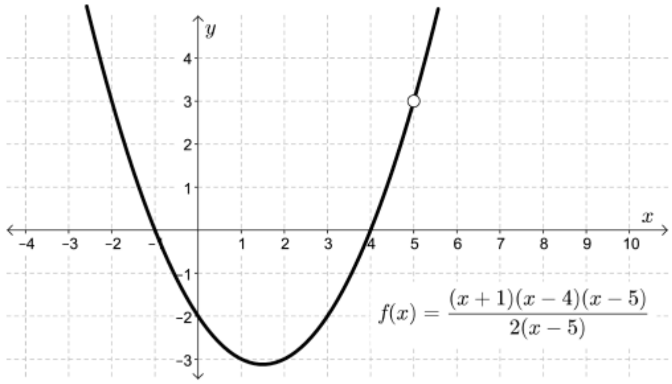


Examples

Example 2: Limits of Functions Having Discontinuities

Let's consider the rational function $f(x) = \dfrac{(x+1)(x-4)(x-5)}{2(x-5)}$.

Since f is defined for all $x \neq 5$, there is a "hole" in the graph at $x = 5$.



Examples

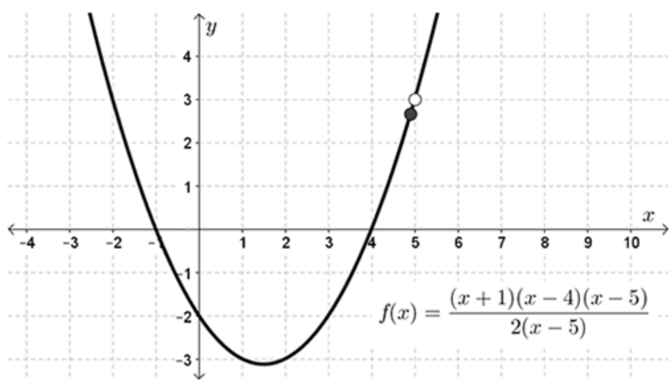
Example 2: Limits of Functions Having Discontinuities

As x approaches 5 from the left side, the limit of $f(x)$ is 3.

$$\lim_{x \to 5^-} f(x) = 3$$

↑

as x appproaches 5 from the left.



Examples

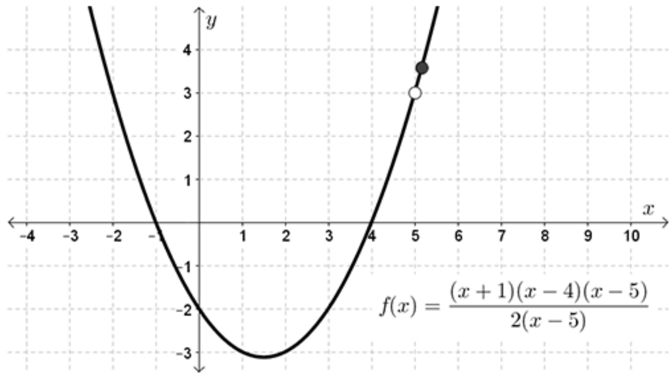
Example 2: Limits of Functions Having Discontinuities

The limit of $f(x)$ as x approaches 5 from the right side is also 3.

$$\lim_{x \to 5^+} f(x) = 3$$

↑

as x appproaches 5 from the right



Examples

Example 2: Limits of Functions Having Discontinuities

In summary, the following three conditions have been met:

$$\lim_{x \to 5^-} f(x) \text{ exists and equals } 3$$

$$\lim_{x \to 5^+} f(x) \text{ exists and equals } 3$$

$$\therefore \lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = 3$$

Thus, $\lim_{x \to 5} f(x)$ exists and equals 3.

Examples

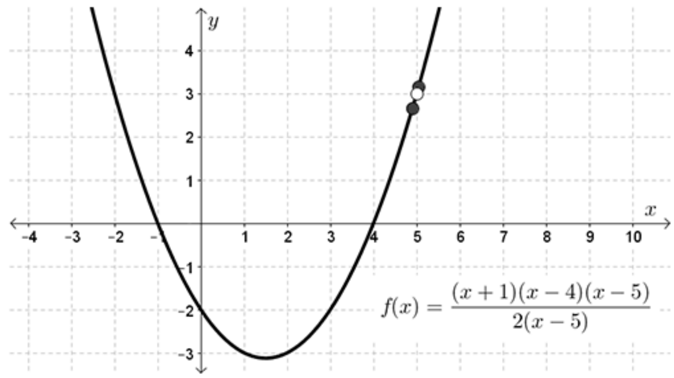
Example 2: Limits of Functions Having Discontinuities

Since $f(x)$ approaches the value 3 from both directions,

$$\lim_{x \to 5} f(x) = 3$$

$\uparrow$

as x appproaches 5



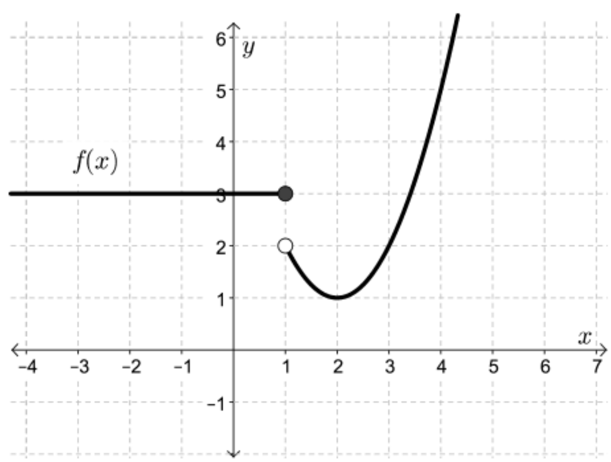
Examples

Example 3: Limits of Piecewise Functions

Let's consider the piecewise function

$$f(x) = \begin{cases} 3 & \text{for } x \leq 1 \\ (x-2)^2 + 1 & \text{for } x > 1 \end{cases}$$

that has a jump discontinuity at $x = 1$.

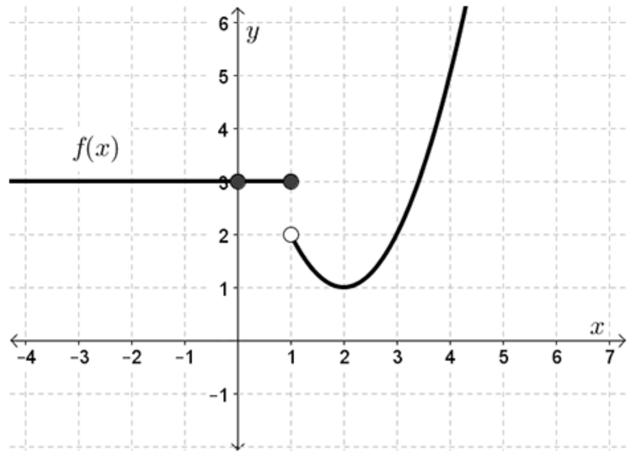


Examples

Example 3: Limits of Piecewise Functions

The limit of $f(x)$ as x approaches 1 from the left side is 3.

$$\lim_{x \to 1^-} f(x) = 3$$

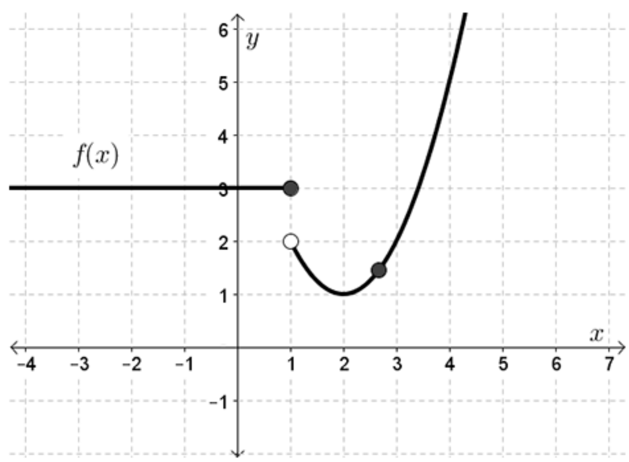


Examples

Example 3: Limits of Piecewise Functions

The limit of $f(x)$ as x approaches 1 from the right side is 2.

$$\lim_{x \to 1^+} f(x) = 2$$



Examples

Example 3: Limits of Piecewise Functions

In summary,

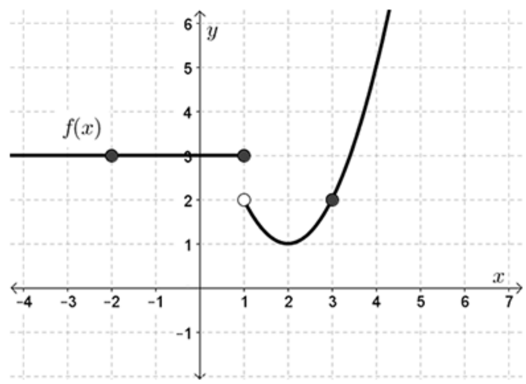
$$\lim_{x \to 1^-} f(x) \text{ exists and equals } 3$$

$$\lim_{x \to 1^+} f(x) \text{ exists and equals } 2$$

$$\therefore \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

Since $f(x)$ does not approach the same value in both directions as x approaches 1, this limit does not exist.

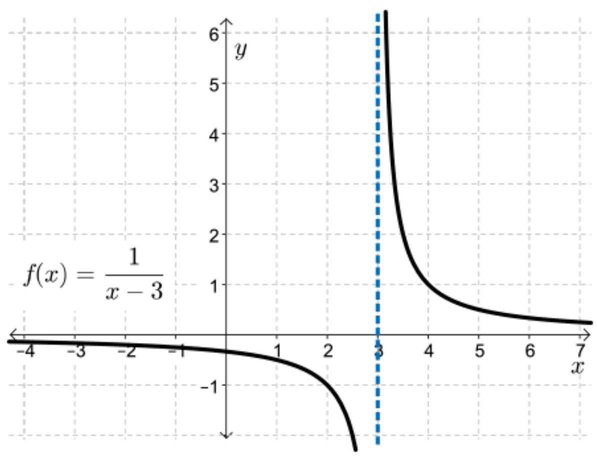
$$\lim_{x \to 1} f(x) \text{ does not exist}$$



Examples

Example 4: Limits of Rational Functions

Let's consider the rational function $f(x) = \dfrac{1}{x-3}$, which has a vertical asymptote of $x = 3$.



Examples

Example 4: Limits of Rational Functions

Before we investigate this limit, let's consider the behaviour of the following important quotients.

Let's think about the value of $g(x) = \frac{1}{x}$ as $x \to 0^-$, by considering the value of the following quotients:

$$g(-0.1) = \frac{1}{-0.1} = -10$$
$$g(-0.01) = \frac{1}{-0.01} = -100$$
$$g(-0.001) = \frac{1}{-0.001} = -1000$$

So, as $x \to 0^-$, $g(x) = \frac{1}{x} \to -\infty$.

Examples

Example 4: Limits of Rational Functions

Similarly, let's think about the value of $g(x) = \frac{1}{x}$ as $x \to 0^+$, by considering the value of the following quotients:

$$g(0.1) = \frac{1}{0.1} = 10$$
$$g(0.01) = \frac{1}{0.01} = 100$$
$$g(0.001) = \frac{1}{0.001} = 1000$$

So, as $x \to 0^+$, $g(x) = \frac{1}{x} \to +\infty$.

Examples

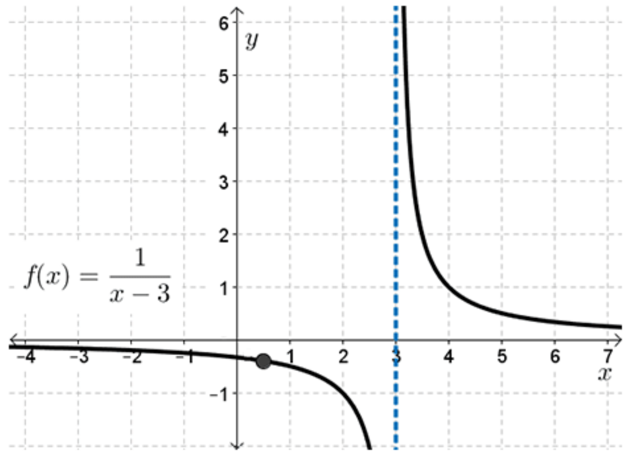
Example 4: Limits of Rational Functions

Now let's apply this knowledge of quotients to the function $f(x) = \frac{1}{x-3}$.

As x approaches 3 from the left side, $f(x) = \frac{1}{\text{a small negative number}}$.

So, $f(x)$ approaches $-\infty$ as x approaches 3 from the left side.

$$\lim_{x \to 3^-} f(x) \to -\infty$$



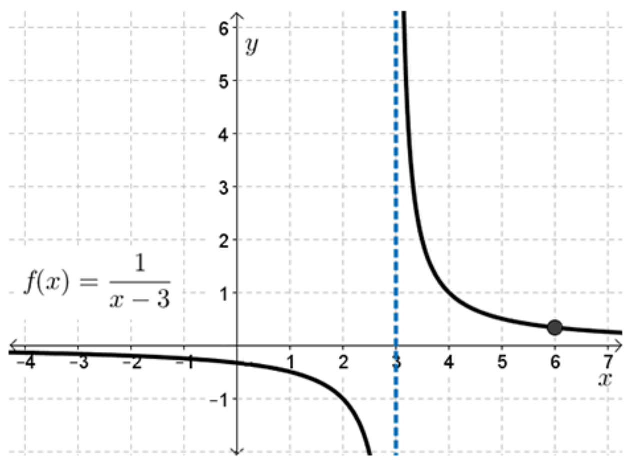
Examples

Example 4: Limits of Rational Functions

Similarly, as x approaches 3 from the right side, $f(x) = \frac{1}{\text{a small positive number}}$.

So, $f(x)$ approaches $+\infty$ as x approaches 3 from the right side.

$$\lim_{x \to 3^+} f(x) \to +\infty$$



Examples

Example 4: Limits of Rational Functions

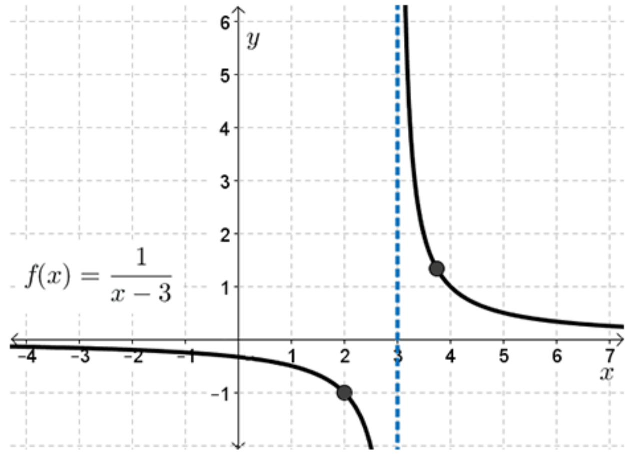
In summary,

$$\lim_{x \to 3^-} f(x) \text{ does not exist}$$

$$\lim_{x \to 3^+} f(x) \text{ does not exist}$$

Since $f(x)$ does not have a left or right side limit, the limit does not exist (DNE).

$$\lim_{x \to 3} f(x) \text{ DNE}$$



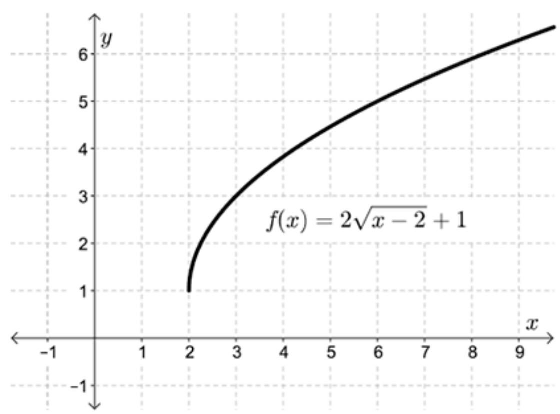
Examples

Example 5: Limits of Functions Containing Radicals

For the function $f(x) = 2\sqrt{x-2} + 1$, evaluate the limits or show that they do not exist:

a. $\lim_{x \to 6} f(x)$

b. $\lim_{x \to 2} f(x)$



Examples

Example 5: Limits of Functions Containing Radicals

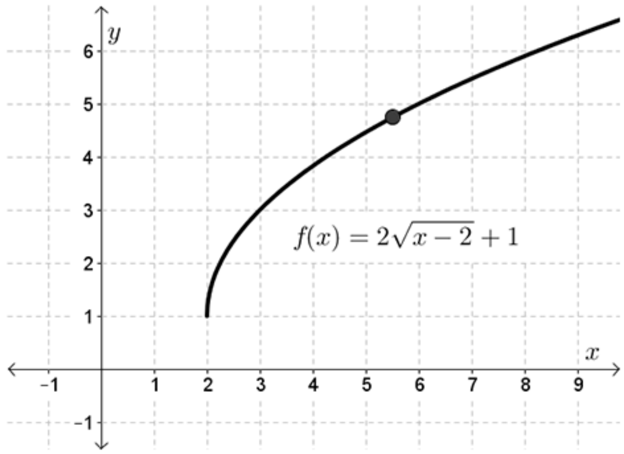
For the function $f(x) = 2\sqrt{x-2}+1$, evaluate the limits or show that they do not exist:

a. $\lim\limits_{x \to 6} f(x)$

As $x \to 6^-$, the value of $x - 2 \to 4$, so $\sqrt{x-2} \to 2$.

Therefore, as x approaches 6 from the left side, the limit of $f(x) = 2\sqrt{x-2}+1$ is 5.

$$\lim_{x \to 6^-} f(x) = 5$$



Examples

Example 5: Limits of Functions Containing Radicals

For the function $f(x) = 2\sqrt{x-2}+1$, evaluate the limits or show that they do not exist:

a. $\lim\limits_{x \to 6} f(x)$

In summary,

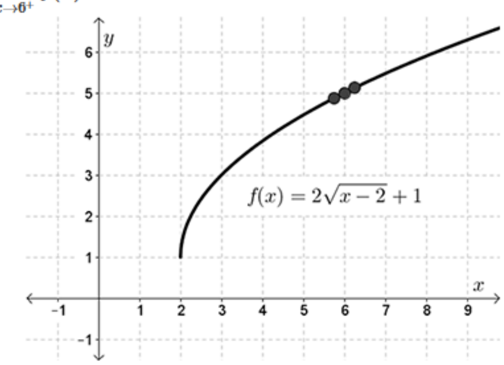
$$\lim_{x \to 6^-} f(x) \text{ exists and equals } 5$$

$$\lim_{x \to 6^+} f(x) \text{ exists and equals } 5$$

$$\therefore \lim_{x \to 6^-} f(x) = \lim_{x \to 6^+} f(x) = 5$$

As x approaches 6 from both directions, $f(x)$ approaches the same value. Therefore, $\lim\limits_{x \to 6} f(x)$ exists and equals 5.

$$\lim_{x \to 6} f(x) = 5$$



Examples

Example 5: Limits of Functions Containing Radicals

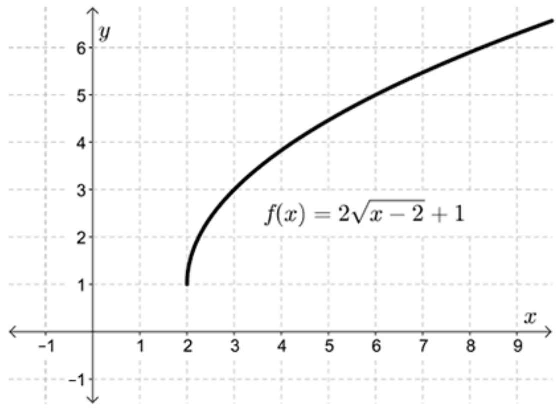
For the function $f(x) = 2\sqrt{x-2}+1$, evaluate the limits or show that they do not exist:

b. $\lim_{x\to 2} f(x)$

As x approaches 2 from the left side, the limit of $f(x) = 2\sqrt{x-2}+1$ does not exist.

The domain of $f(x)$ is $x \geq 2$, which means that we cannot approach the value $x = 2$ from the left side because $f(x)$ is undefined for all $x < 2$.

$$\lim_{x\to 2^-} f(x) \text{ DNE}$$



Examples

Example 5: Limits of Functions Containing Radicals

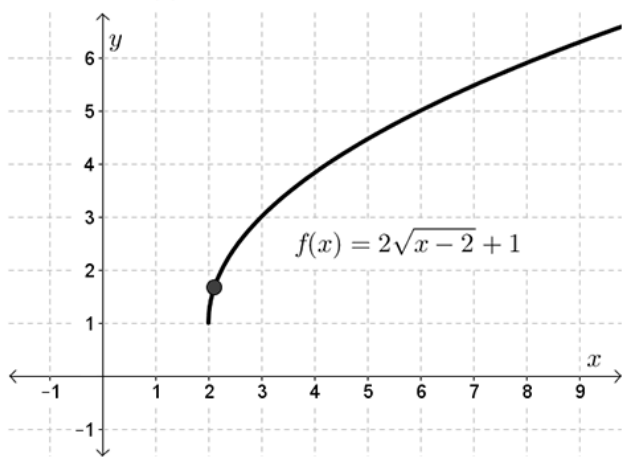
For the function $f(x) = 2\sqrt{x-2}+1$, evaluate the limits or show that they do not exist:

b. $\lim_{x\to 2} f(x)$

As $x \to 2^+$, the value of $x - 2 \to 0$, so $\sqrt{x-2} \to 0$.

Therefore, as x approaches 2 from the right side, the limit of $f(x) = 2\sqrt{x-2}+1$ is 1.

$$\lim_{x\to 2^+} f(x) = 1$$



Examples

Example 5: Limits of Functions Containing Radicals

For the function $f(x) = 2\sqrt{x-2}+1$, evaluate the limits or show that they do not exist:

b. $\lim_{x \to 2} f(x)$

In summary,

$$\lim_{x \to 2^-} f(x) \text{ does not exist}$$

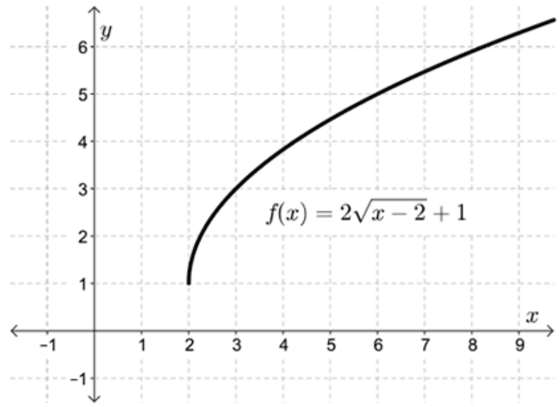
$$\lim_{x \to 2^+} f(x) \text{ exists and equals } 1$$

Since $\lim_{x \to 2^-} f(x)$ does not exist, $\lim_{x \to 2} f(x)$ does not exist.

$$\lim_{x \to 2} f(x) \text{ DNE}$$

Note

It is common to say that $\lim_{x \to 2} 2\sqrt{x-2}+1$ does in fact exist. Since the interval $x < 2$ is not in the domain of the function, we may ignore the left-sided limit entirely. (That is, we ignore the direction which is not in the domain).



Examples

Challenge Problem

A piecewise defined function is given as

$$f(x) = \begin{cases} ax+b & x < 0 \\ x^2+3 & 0 \le x \le 3 \\ -a(x-5)^2 & x > 3 \end{cases}$$

If the limit exists at both $x = 0$ and $x = 3$, find the values of a and b.

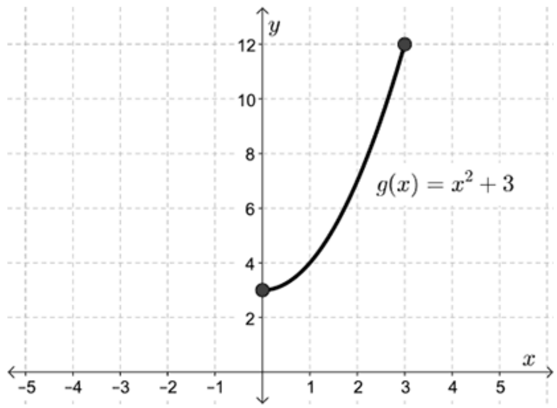
Examples

Challenge Problem

$$f(x) = \begin{cases} ax+b & x<0 \\ x^2+3 & 0 \le x \le 3 \\ -a(x-5)^2 & x>3 \end{cases}$$

Solution

For $0 \le x \le 3$, we know the shape of the curve of $f(x)$.
Let's examine the graph of this section of $f(x)$.



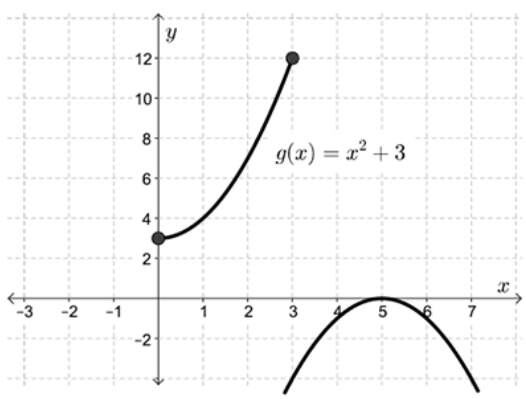
Examples

Challenge Problem

$$f(x) = \begin{cases} ax+b & x<0 \\ x^2+3 & 0 \le x \le 3 \\ -a(x-5)^2 & x>3 \end{cases}$$

Solution

Now $f(x) = -a(x-5)^2$ for $x > 3$.
Suppose $a = 1$. We have $f(x) = -(x-5)^2$. This quadratic is negative so it's a downward facing parabola with vertex $(5, 0)$.



Examples

Challenge Problem

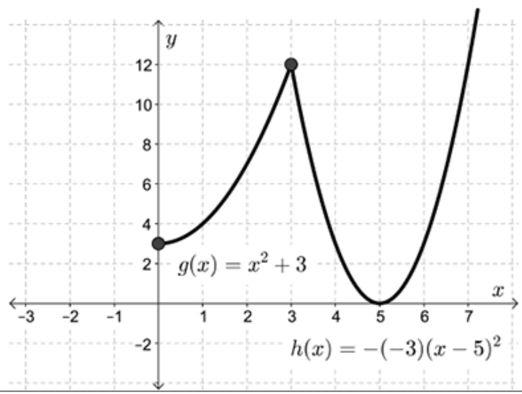
$$f(x) = \begin{cases} ax + b & x < 0 \\ x^2 + 3 & 0 \le x \le 3 \\ -a(x-5)^2 & x > 3 \end{cases}$$

Solution

When $x = 3$, the value of $y = -1(x-5)^2 = -4$.

For the limit to exist at $x = 3$, we need an a-value that will map the point $(3, -4)$ to the point $(3, 12)$.

This can be achieved by making $a = -3$.



Examples

Challenge Problem

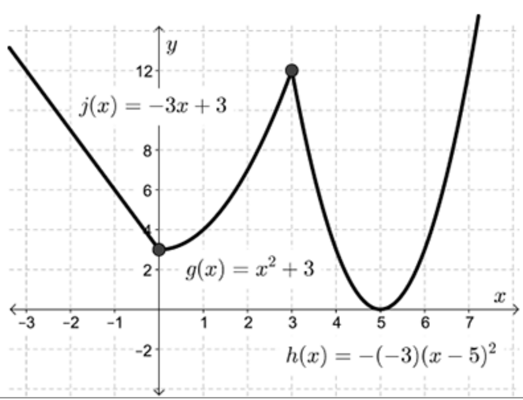
$$f(x) = \begin{cases} ax + b & x < 0 \\ x^2 + 3 & 0 \le x \le 3 \\ -a(x-5)^2 & x > 3 \end{cases}$$

Solution

Since $a = -3$, we know that for $x < 0$, $f(x) = -3x + b$.

For the limit to exist at $x = 0$, we need a value of b that will cause this line to have a y-intercept of 3.

Thus, $b = 3$.



Examples

Challenge Problem

Solution

Therefore, the limit of the piecewise defined function

$$f(x) = \begin{cases} ax + b & x < 0 \\ x^2 + 3 & 0 \le x \le 3 \\ -a(x-5)^2 & x > 3 \end{cases}$$

exists at both $x = 0$ and $x = 3$, when $a = -3$ and $b = 3$.