Limits at Infinity

Evaluating Limits of Convergent Sequences

A **sequence** is a function whose domain is the set of positive integers $n = 1, 2, 3, \ldots$.

The values or individual terms of a sequence are generally denoted by a subscript of n on t. In other words, we use t_n rather than $f(n)$.
For example, the list of all positive odd numbers forms the sequence $1, 3, 5, 7, \ldots$.
This sequence could be represented algebraically by two different formulas:

1. Recursive Formula

$t_n = t_{n-1} + 2$ where $t_1 = 1$.

2. Arithmetic Formula

$$\begin{aligned} t_n &= 1 + 2(n-1) \\ &= 2n - 1 \text{ for } n = 1, 2, 3, \ldots \end{aligned}$$

If we continue this sequence of numbers, would this sequence approach a single value?
In other words, as $n \to +\infty$ does t_n approach a limit?
As n increases, we see that t_n becomes arbitrarily large in value.
Therefore, as $n \to +\infty$, $t_n \to +\infty$.
We could use limits to write this as

$$\lim_{n \to +\infty} (2n - 1) \to +\infty$$

Examples

Example 1

Consider the sequence defined by $t_n = \dfrac{1}{n^2 + 4}$.

a. List the first five terms of this sequence.

b. Evaluate t_{10}, t_{100}, and t_{1000}. Express your answers as fractions and as decimals.

c. Using your answers from part **b.**, predict the value of $\lim\limits_{n \to \infty} \dfrac{1}{n^2 + 4}$.

Solution

a. The first five terms of the sequence are

t_1	t_2	t_3	t_4	t_5	$\ldots$
$\frac{1}{1^2+4}$	$\frac{1}{2^2+4}$	$\frac{1}{3^2+4}$	$\frac{1}{4^2+4}$	$\frac{1}{5^2+4}$	$\ldots$
$\frac{1}{5}$	$\frac{1}{8}$	$\frac{1}{13}$	$\frac{1}{20}$	$\frac{1}{29}$	$\ldots$

Examples

Example 1

Consider the sequence defined by $t_n = \dfrac{1}{n^2+4}$.

a. List the first five terms of this sequence.

b. Evaluate t_{10}, t_{100}, and t_{1000}. Express your answers as fractions and as decimals.

c. Using your answers from part **b.**, predict the value of $\lim\limits_{n\to\infty} \dfrac{1}{n^2+4}$.

Solution

b. $t_{10} = \dfrac{1}{10^2+4} = \dfrac{1}{104} \approx 0.0096$

$t_{100} = \dfrac{1}{100^2+4} = \dfrac{1}{10004} \approx 0.00009996$

$t_{1000} = \dfrac{1}{1000^2+4} = \dfrac{1}{1000004} \approx 0.000000999996$

Examples

Example 1

Consider the sequence defined by $t_n = \dfrac{1}{n^2+4}$.

a. List the first five terms of this sequence.

b. Evaluate t_{10}, t_{100}, and t_{1000}. Express your answers as fractions and as decimals.

c. Using your answers from part **b.**, predict the value of $\lim\limits_{n\to\infty} \dfrac{1}{n^2+4}$.

Solution

c. In this example, we see that as $n \to \infty$, t_n appears to approach the single value 0.

We can justify saying that $\lim\limits_{n\to\infty} \dfrac{1}{n^2+4} = 0$ by noting that we can make $\dfrac{1}{n^2+4}$ as near as we like to 0 by making n sufficiently large.

Sequences that approach a single finite value are called convergent.

Examples

Example 2

Consider the sequence defined by $t_n = \dfrac{n-3}{n}$.

a. List the first five terms of this sequence.

b. Evaluate t_{10}, t_{100} and t_{1000}. Express your answers as fractions and as decimals.

c. Using your answers from part **b.**, predict the value of $\lim\limits_{n\to\infty} \dfrac{n-3}{n}$.

Solution

a. The first five terms of the sequence are

t_1	t_2	t_3	t_4	t_5	...
$\frac{1-3}{1}$	$\frac{2-3}{2}$	$\frac{3-3}{3}$	$\frac{4-3}{4}$	$\frac{5-3}{5}$	...
-2	$-\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{2}{5}$	...

Examples

Example 2

Consider the sequence defined by $t_n = \dfrac{n-3}{n}$.

a. List the first five terms of this sequence.

b. Evaluate t_{10}, t_{100} and t_{1000}. Express your answers as fractions and as decimals.

c. Using your answers from part **b.**, predict the value of $\lim\limits_{n\to\infty} \dfrac{n-3}{n}$.

Solution

b. $t_{10} = \dfrac{10-3}{10} = \dfrac{7}{10} = 0.7$

$t_{100} = \dfrac{100-3}{100} = \dfrac{97}{100} = 0.97$

$t_{1000} = \dfrac{1000-3}{1000} = \dfrac{997}{1000} = 0.997$

Examples

Example 2

Consider the sequence defined by $t_n = \dfrac{n-3}{n}$.

a. List the first five terms of this sequence.

b. Evaluate t_{10}, t_{100} and t_{1000}. Express your answers as fractions and as decimals.

c. Using your answers from part **b.**, predict the value of $\lim\limits_{n\to\infty} \dfrac{n-3}{n}$.

Solution

c. Thus, $\lim\limits_{n\to\infty} \dfrac{n-3}{n}$ appears to be 1.

To understand why this occurs, note that $\dfrac{n-3}{n} = 1 - \dfrac{3}{n}$, and since $\lim\limits_{n\to\infty} \dfrac{3}{n} = 0$, the sequence t_n has limit 1.

Examples

Example 3

Consider the sequence defined by $t_n = \left(\dfrac{1-n}{n}\right)^n$.

a. List the first five terms of this sequence.

b. Graph these values.

c. Using your answers from part **b.**, predict the value of $\lim\limits_{n\to\infty} \left(\dfrac{1-n}{n}\right)^n$.

Solution

a. The first five terms of the sequence are

t_1	t_2	t_3	t_4	t_5	...
$\left(\frac{1-1}{1}\right)^1$	$\left(\frac{1-2}{2}\right)^2$	$\left(\frac{1-3}{3}\right)^3$	$\left(\frac{1-4}{4}\right)^4$	$\left(\frac{1-5}{5}\right)^5$	...
0	$\frac{1}{4}$	$-\frac{8}{27}$	$\frac{81}{256}$	$-\frac{1024}{3125}$	...

Examples

Example 3

Consider the sequence defined by $t_n = \left(\frac{1-n}{n}\right)^n$.

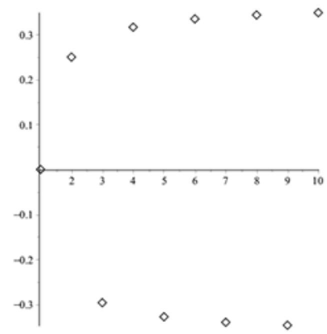
a. List the first five terms of this sequence.

b. Graph these values.

c. Using your answers from part **b.**, predict the value of $\lim_{n\to\infty}\left(\frac{1-n}{n}\right)^n$.

Solution

b.



Examples

Example 3

Consider the sequence defined by $t_n = \left(\frac{1-n}{n}\right)^n$.

a. List the first five terms of this sequence.

b. Graph these values.

c. Using your answers from part **b.**, predict the value of $\lim_{n\to\infty}\left(\frac{1-n}{n}\right)^n$.

Solution

c. As we see from the graph, the values of this sequence alternate between positive values and negative values.
This sequence is not approaching a single finite value as $n \to \infty$.
Since a single value is not approached, we say that the limit does not exist and that the sequence is divergent.

$$\lim_{n\to\infty}\left(\frac{1-n}{n}\right)^n \text{ does not exist.}$$

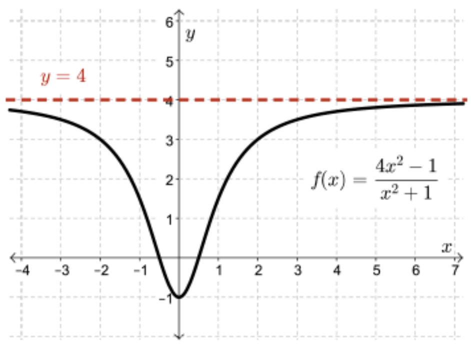
Connecting Limits at Infinity with Horizontal Asymptotes

Horizontal asymptotes affect the end behaviour of a graph.
The end behaviour can be studied by considering very large positive and negative values of x and their corresponding value of $f(x)$.
As we consider larger and larger positive values of x, we say that $x \to +\infty$.
As we consider larger and larger negative values of x, we say that $x \to -\infty$.

Examples

Example 4

Let's consider the graph of the function $f(x) = \dfrac{4x^2 - 1}{x^2 + 1}$.



We see that as $x \to +\infty$, and as $x \to -\infty$, $f(x) \to 4$.
How can we use the algebraic representation of the function to determine this horizontal asymptote?

Examples

Example 4

Let's consider the graph of the function $f(x) = \dfrac{4x^2 - 1}{x^2 + 1}$.

How can we use the algebraic representation of the function to determine this horizontal asymptote?

Solution

Consider $\lim\limits_{x \to \infty} \dfrac{1}{x^n}$ for any positive number n.
As $x \to \infty$, $\dfrac{1}{x^n} \to 0$.
This is a very special limit, which we will use to determine horizontal asymptotes.
To find the horizontal asymptote of $f(x) = \dfrac{4x^2 - 1}{x^2 + 1}$, begin by identifying the highest power of x within this function and divide each term by this power.

$$\lim_{x \to \infty} \frac{4x^2 - 1}{x^2 + 1} = \lim_{x \to \infty} \left(\frac{4x^2 - 1}{x^2 + 1}\right) \left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right) = \lim_{x \to \infty} \frac{\frac{4x^2}{x^2} - \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{1}{x^2}}$$

Now, simplify each term.

$$\lim_{x \to \infty} \frac{4 - \frac{1}{x^2}}{1 + \frac{1}{x^2}} \text{ (for } x \neq 0)$$

Examples

Example 4

Let's consider the graph of the function $f(x) = \dfrac{4x^2 - 1}{x^2 + 1}$.

How can we use the algebraic representation of the function to determine this horizontal asymptote?

Solution

Identify the terms where $\lim\limits_{x \to \infty} \dfrac{1}{x^n} = 0$.
Substitute 0 for each of these terms and evaluate the limit.

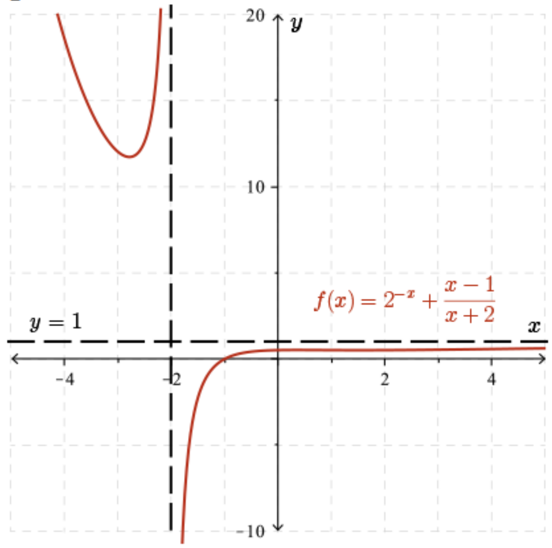
$$\lim_{x \to \infty} \frac{4 - \frac{1}{x^2}}{1 + \frac{1}{x^2}} = \frac{4 - 0}{1 + 0} = 4$$

Note that since $f(x) = \dfrac{4x^2 - 1}{x^2 + 1}$ is an even function, the limit is the same whether $x \to +\infty$ or $x \to -\infty$, so $y = 4$ is a horizontal asymptote in both cases.

Examples

Example 5

Consider the graph of the function $f(x) = 2^{-x} + \frac{x-1}{x+2}$. A portion of its graph is shown.



We note the following properties of the function $f(x)$ (that are suggested by the given graph and can be verified algebraically):

As $x \to \infty$, $f(x) \to 1$.
It follows that the line $y = 1$ is a horizontal asymptote of the graph $y = f(x)$.

As $x \to -\infty$, $f(x) \to \infty$.
Therefore, $f(x)$ does not have a horizontal asymptote in the other direction.
We see that a horizonal asymptote does not need to occur in both directions.
$f(x)$ does not necessarily approach a horizontal asympotote as $x \to +\infty$ **and** as $x \to -\infty$.

Examples

Challenge Problem

Try this on your own and then click play to see the solution.

A rational function has a horizontal asymptote at $y = \frac{1}{2}$ and the degree of its numerator and denominator is greater than or equal to 1. Determine a possible equation for this function.

Solution

In order for a rational function to have a non-zero horizontal asymptote, the degree of the numerator must be equal to the degree of the denominator.

Then, the rational function will have a horizontal asymptote at $y = \frac{a}{b}$, where a and b are the coefficients of the highest degree term in the numerator and denominator, respectively.

For the function to have a horizontal asymptote at $y = \frac{1}{2}$, the ratio of the leading coefficient of the numerator to the leading coefficient of the denominator must be $\frac{1}{2}$.

One such function that satisfies these conditions is $y = \frac{x+1}{2x}$.