Integration by Substitution

In This Module

- We will learn one of the two main methods of integration: the method of substitution.

The second method is called integration by parts, and it will be covered in the next module.

As we have seen, every differentiation rule gives rise to a corresponding integration rule. The method of substitution arises from the chain rule for differentiation.

It is often the case that a simple change of variables can change an unfamiliar integral into an integral found in our table of formulas.
Roughly speaking, the method of substitution is used when the integrand involves a composition of functions and a change of variables can be used to simplify the integrand.

Examples

Example 1

Determine $\int 2x\sqrt{x^2+1}\,dx$.

Solution

This integrand cannot be found in the table of known antiderivatives, so we need a new method to evaluate this integral.
We will proceed with a change of variables to simplify this integral.
With a change of variables $u = x^2+1$, we can rewrite $\sqrt{x^2+1}$ as $\sqrt{u}$.
This function can be integrated with respect to u using the power rule.
Recall from our table that

$$\int \sqrt{u}\,du = \int u^{\frac{1}{2}}\,du = \frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + c = \frac{2}{3}\,u^{\frac{3}{2}} + c$$

Let's ignore the factor of $2x$ for a moment and let's see what happens when we differentiate
$g\Big(u(x)\Big) = \frac{2}{3}\left(u(x)\right)^{\frac{3}{2}} = \frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}}$ with respect to x, using the chain rule.

Examples

Example 1

Determine $\int 2x\sqrt{x^2+1}\,dx$.

Solution

$$\begin{aligned}
\frac{d}{dx}\Big(g(u(x))\Big) &= \frac{d}{du}\Big(g(u)\Big)\left(\frac{du}{dx}\right) && \text{by the chain rule}\\
&= \frac{d}{du}\left(\frac{2}{3}u^{\frac{3}{2}}\right)\frac{d}{dx}\left(x^2+1\right)\\
&= \left(\frac{2}{3}\right)\left(\frac{3}{2}u^{\frac{1}{2}}\right)(2x)\\
&= \left(\sqrt{u}\right)(2x)\\
&= \left(\sqrt{x^2+1}\right)(2x) && \text{as } u = x^2+1\\
&= 2x\sqrt{x^2+1}
\end{aligned}$$

Since $\frac{d}{dx}\left(\frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}}+c\right) = 2x\sqrt{x^2+1}$, we have verified the following indefinite integral

$$\int 2x\sqrt{x^2+1}\,dx = \frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}}+c$$

Examples

Example 1

Determine $\int 2x\sqrt{x^2+1}\,dx$.

Remark

The solution works here for the same reason as our method for integrating $\int \sqrt{x+1}\,dx$ at the end of the previous module.

We can think of $\int \sqrt{x+1}\,dx$ as $\int (1)\left(\sqrt{x+1}\right)\,dx$ where 1 is the derivative of $u = x+1$ with respect to x.

Similarly, we have $\int 2x\sqrt{x^2+1}\,dx$ where $2x$ is the derivative of $u = x^2+1$ with respect to x.

This suggests a general method for dealing with any integral of the form

$$\int \left(\frac{du}{dx}\right)\Big(f(u)\Big)\,dx$$

for any function $f(u)$ that appears in our table of known integrals.

The Method of Substitution

We will introduce the method of substitution using the following example. The formal rule will follow.

Example 2

Determine $\displaystyle\int \frac{3x^2}{\sqrt{x^3-2}}\ dx$.

Solution

First, note that we have not yet seen an antiderivative of the function $f(x) = \dfrac{3x^2}{\sqrt{x^3-2}}$.

Second, observe that we cannot simplify this integrand in any meaningful way.

We will proceed with the method used in example 1.

If we let $u = x^3 - 2$, then we have $\dfrac{du}{dx} = 3x^2$, and we will rewrite our integral as follows:

$$\int \frac{3x^2}{\sqrt{x^3-2}}\ dx = \int \frac{1}{\sqrt{x^3-2}}\left(3x^2\right)\ dx$$

We will abuse the notation for a moment and rewrite the equation $\dfrac{du}{dx} = 3x^2$ as $du = 3x^2\ dx$, treating the term $\dfrac{du}{dx}$ as a fraction of differentials.

Notice that the term $3x^2\ dx$ "appears" in our integral.

The Method of Substitution

We will introduce the method of substitution using the following example. The formal rule will follow.

Example 2

Determine $\displaystyle\int \frac{3x^2}{\sqrt{x^3-2}}\ dx$.

Solution

Recall, $u = x^3 - 2$ and $du = 3x^2 dx$.

Next, we will attempt to rewrite the entire integral in terms of the new variable, u.

In other words, we are substituting the new variable u for our old variable of integration, x.

$$\begin{aligned}
\int \frac{3x^2}{\sqrt{x^3-2}}\ dx &= \int \underbrace{\left(\frac{1}{\sqrt{x^3-2}}\right)}_{1/\sqrt{u}} \underbrace{\left(3x^2\ dx\right)}_{du} \\
&= \int \frac{1}{\sqrt{u}}\ du = \int u^{-\frac{1}{2}}\ du \\
&= \frac{u^{-\frac{1}{2}+1}}{\frac{-1}{2}+1} + c \\
&= 2u^{\frac{1}{2}} + c && \text{using the known indefinite integral} \\
&= 2\sqrt{x^3-2} + c && \text{substituting } u = x^3 - 2
\end{aligned}$$

The Method of Substitution

We will introduce the method of substitution using the following example. The formal rule will follow.

Example 2

Determine $\displaystyle\int \frac{3x^2}{\sqrt{x^3-2}}\,dx$.

Solution

Now we need to check, using the chain rule, that our approach led us to the correct indefinite integral.

$$\frac{d}{dx}\left(2\sqrt{x^3-2}+c\right) = 2\left(\frac{1}{2}\right)(x^3-2)^{-\frac{1}{2}}(3x^2)+0 = \frac{3x^2}{\sqrt{x^3-2}}$$

and therefore our answer is correct.

The Method of Substitution

Summary

If an integral is of the form $\displaystyle\int f(g(x))g'(x)\,dx$ for some functions f and g, then by substituting $u=g(x)$, we can rewrite the integral as $\displaystyle\int f(u)\,du$, effectively eliminating the composition of functions. More formally,

Substitution Rule

If $u=g(x)$, then

$$\int f\Big(g(x)\Big)g'(x)\,dx = \int f(u)\,du$$

Integrating f with respect to u will, of course, produce a function in terms of u. Don't forget to plug in $u=g(x)$ at the end to get back to a function of x.

A formal proof that this rule is correct is not difficult, and it only requires the chain rule and the definition of an antiderivative. You will be asked to do this in the exercises.

Examples

Example 3

Rewrite the following integrals using the substitution rule and then integrate.

a. $\int e^{7x}\,dx$ **b.** $\int \frac{3x}{\cos^2(5x^2+2)}\,dx$

Solution

a. Here $u = 7x$, so $\frac{du}{dx} = 7$ and $du = 7dx$. Substituting our variable u for the variable x gives the following:

$$\begin{aligned}\int \underbrace{e^{7x}}_{e^u}\,\underbrace{dx}_{du/7} &= \int (e^u)\left(\frac{1}{7}\,du\right)\\ &= \frac{1}{7}\int e^u\,du\\ &= \frac{1}{7}\left[e^u + c\right]\\ &= \frac{1}{7}e^u + \frac{1}{7}c\\ &= \frac{1}{7}e^u + C\\ &= \frac{1}{7}e^{7x} + C \quad \text{as } u = 7x\end{aligned}$$

Examples

Example 3

Rewrite the following integrals using the substitution rule and then integrate.

a. $\int e^{7x}\,dx$ **b.** $\int \frac{3x}{\cos^2(5x^2+2)}\,dx$

Solution

b. We have $\int \frac{3x}{\cos^2(5x^2+2)}\,dx = \int 3x\sec^2(5x^2+2)\,dx$, and we know that we have a formula for $\int \sec^2(u)\,du$.

This suggests letting $u = 5x^2 + 2$.

Then we have $\frac{du}{dx} = 10x$ or $x\,dx = \frac{1}{10}\,du$.

Examples

Example 3

Rewrite the following integrals using the substitution rule and then integrate.

a. $\int e^{7x}\,dx$ **b.** $\int \frac{3x}{\cos^2(5x^2+2)}\,dx$

Solution

Substituting our new variable u, we have

$$\begin{aligned}
3\int x\sec^2(5x^2+2)\,dx &= 3\int \underbrace{\sec^2(5x^2+2)}_{\sec^2(u)}\,\underbrace{x\,dx}_{\frac{1}{10}\,du} \\
&= 3\int \sec^2(u)\,\frac{1}{10}\,du \\
&= \frac{3}{10}\int \sec^2(u)\,du \\
&= \frac{3}{10}\tan(u)+c \\
&= \frac{3}{10}\tan(5x^2+2)+c
\end{aligned}$$

Note that, here, we skipped the step of setting $C = \frac{3}{10}\,c$ and just went straight to the final constant, c.

Substitution for Definite Integrals

How do we apply the method of substitution to evaluate definite integrals?

For example, how do we evaluate the integral

$$\int_0^1 2x\sqrt{x^2+1}\,dx$$

Here we have two choices. We have already seen how to evaluate indefinite integrals using substitution, so we can use this method to find an antiderivative of the integrand $2x\sqrt{x^2+1}$ and then apply the fundamental theorem as usual. Alternatively, we can transform the entire integral, including the limits of integration, using a change of variables and evaluate a new definite integral of the form $\int_c^d f(u)\,du$.

Examples

Example 4

Evaluate $\int_0^1 2x\sqrt{x^2+1}\,dx$ using the method of substitution.

Solution

Method 1

Recall that $\int 2x\sqrt{x^2+1}\,dx = \frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}} + c$, so we can apply the fundamental theorem using the simplest antiderivative $\frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}}$.

$$\begin{aligned}\int_0^1 2x\sqrt{x^2+1}\,dx &= \left[\frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}}\right]_0^1 \\ &= \frac{2}{3}\left((1)^2+1\right)^{\frac{3}{2}} - \frac{2}{3}\left((0)^2+1\right)^{\frac{3}{2}} \\ &= \frac{2}{3}(2)^{\frac{3}{2}} - \frac{2}{3}(1)^{\frac{3}{2}} \\ &= \frac{2}{3}\left(\sqrt{8}-1\right) \\ &= \frac{4}{3}\sqrt{2} - \frac{2}{3}\end{aligned}$$

Examples

Example 4

Evaluate $\int_0^1 2x\sqrt{x^2+1}\,dx$ using the method of substitution.

Solution

Method 2

Recall that we have $u = x^2+1$ and $du = 2x\,dx$.

Let's perform a change of variables as before, but now we also include a change in the limits of integration.

$$\begin{aligned}\int_0^1 2x\sqrt{x^2+1}\,dx &= \int_c^d \sqrt{u}\,du \\ x=0 \text{ to } x=1 \qquad & \qquad u=c \text{ to } u=d\end{aligned}$$

Observe that when $x=0$, $u=(0)^2+1=1$ and when $x=1$, $u=(1)^2+1=2$. Therefore, we get

$$\begin{aligned}\int_0^1 2x\sqrt{x^2+1}\,dx &= \int_1^2 \sqrt{u}\,du \\ &= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^2 \\ &= \frac{2}{3}(2)^{\frac{3}{2}} - \frac{2}{3}(1)^{\frac{3}{2}} \\ &= \frac{4}{3}\sqrt{2} - \frac{2}{3}\end{aligned}$$

Examples

Example 4

Evaluate $\int_0^1 2x\sqrt{x^2+1}\,dx$ using the method of substitution.

Remark

While both methods are correct, we will often choose method 2 simply because it is quicker.
The first method requires us to go from the variable x to the variable u, and from u back again to x.
The second method requires us to compute a second set of limits of integration, but we can evaluate the integral without passing back to x.

Substitution Rule for Definite Integrals

If $u = g(x)$, then

$$\int_a^b f\Big(g(x)\Big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Examples

Example 5

Evaluate the definite integral $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos(x)\sin(x)\,dx$.

Solution

It may not be immediately clear how to use substitution here.

Remember that, roughly speaking, we are looking for a function $f(u)$ along with the derivative $\frac{du}{dx}$, together in the integrand.

If we let $u = \sin(x)$, then we have $\frac{du}{dx} = \cos(x)$ or $du = \cos(x)\,dx$.
You could also pick $u = \cos(x)$ and then $du = -\sin(x)\,dx$.

Examples

Example 5

Evaluate the definite integral $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos(x)\sin(x)\,dx$.

Solution

Using method 2 we have $u = \sin(x)$. When $x = \frac{\pi}{6}$, $u = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and when $x = \frac{\pi}{2}$, $u = \sin\left(\frac{\pi}{2}\right) = 1$.

Therefore, we have

$$\begin{aligned}
\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos(x)\sin(x)\,dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \underbrace{\sin(x)}_{u}\,\underbrace{\cos(x)\,dx}_{du} \\
&= \int_{\frac{1}{2}}^{1} u\,du \qquad \text{now integrating from } u = \frac{1}{2} \text{ to } u = 1 \\
&= \left[\frac{u^2}{2}\right]_{\frac{1}{2}}^{1} \\
&= \frac{1}{2}\left((1)^2 - \left(\frac{1}{2}\right)^2\right) = \frac{3}{8}
\end{aligned}$$

Check your answer using method 1!

Examples

Example 6

Determine $\int \tan(x)\,dx$.

Solution

We have $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$, so now we have some options for a substitution.

We will try the substitution $u = \cos(x)$ as we have the derivative of $\cos(x)$ appearing in the numerator.

Why can't we use also the substitution $u = \sin(x)$ this time?

If $u = \sin(x)$ then $du = \cos(x)\,dx$, which does not appear in the integrand. We have the related expression $\frac{1}{\cos(x)}\,dx$, but it is not clear how to write this in terms of our new variable u.

Examples

Example 6

Determine $\int \tan(x)\,dx$.

Solution

If $u = \cos(x)$, then $du = -\sin(x)\,dx$, so we have

$$\begin{aligned}\int \frac{\sin(x)}{\cos(x)}\,dx &= \int \underbrace{\frac{1}{\cos(x)}}_{1/u}\underbrace{\sin(x)\,dx}_{-du} \\ &= \int \frac{1}{u}\,(-du) \\ &= -\int \frac{1}{u}\,du \\ &= -\ln|u| + c \\ &= -\ln|\cos(x)| + c\end{aligned}$$

Examples

Example 6

Determine $\int \tan(x)\,dx$.

Solution

Since $-\ln|\cos(x)| = \ln\left(|\cos(x)|^{-1}\right) = \ln\left|\frac{1}{\cos(x)}\right| = \ln|\sec(x)|$, we often write this integration formula as follows:

$$\int \tan(x)\,dx = \ln|\sec(x)| + c$$

Summary

Substitution Rule

If $u = g(x)$, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

The method of substitution will be successful if the integral can be decomposed as $\int f(g(x))g'(x)\,dx$ and an antiderivative of f is known.

Some examples include

- $\int 2x\sqrt{x^2+1}\,dx$ with $f(x) = \sqrt{x}$, $g(x) = x^2+1$, and $g'(x) = 2x$;
- $\int x\sqrt{x^2+1}\,dx = \int \frac{1}{2}\sqrt{x^2+1}(2x)\,dx$ with $f(x) = \frac{1}{2}\sqrt{x}$, $g(x) = x^2+1$, and $g'(x) = 2x$; and
- $\int 3x\sqrt{x^2+1}\,dx = \frac{3}{2}\int 2x\sqrt{x^2+1}\,dx$ with $f(x) = \sqrt{x}$, $g(x) = x^2+1$, and $g'(x) = 2x$.

Summary

Substitution Rule

If $u = g(x)$, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

What if the integrand cannot be decomposed as noted previously? Can we use the method of substitution to do the following:

- Determine $\int 2x^3\sqrt{x^2+1}\,dx$.

  Let $u = x^2+1$, then $du = 2x\,dx$.
  What do we do with the extra factor of x^2 in the integrand?
  Notice that $x^2 = u-1$.

- Determine $\int 2x^2\sqrt{x^2+1}\,dx$.

  Let $u = x^2+1$, then $du = 2x\,dx$.
  What do we do with the extra factor of x in the integrand?
  Notice that $x = \pm\sqrt{u-1}$.

Summary

Substitution Rule

If $u = g(x)$, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

What if the integrand cannot be decomposed as noted previously? Can we use the method of substitution to do the following:

- Determine $\int \cos(x)\ln(\sin(x))\,dx$.

  Let $u = \sin(x)$, then $du = \cos(x)\,dx$.

  Then the integral becomes $\int \cos(x)\ln(\sin(x))\,dx = \int \ln(u)\,du$.

  This is certainly a simpler integrand, but do we know an antiderivative of $f(u) = \ln(u)$?

Summary

Recall six of our examples involving compositions of functions.

- $\int 2x\sqrt{x^2+1}\,dx = \int \sqrt{u}\,du$
- $\int x\sqrt{x^2+1}\,dx$
- $\int 3x\sqrt{x^2+1}\,dx$
- $\int 2x^3\sqrt{x^2+1}\,dx$
- $\int 2x^2\sqrt{x^2+1}\,dx$
- $\int \cos(x)\ln(\sin(x))\,dx = \int \ln(u)\,du$

Final Thought

We would like to know an antiderivate of $\ln(x)$.

None of our current methods will work.

We will learn a new method, called integration by parts, the next module.