The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Introduction to Related Rates

Introduction

In our earlier discussions of rates of change, we only dealt with the rate of change of a single quantity at a time.

In many physical situations, examining the relationship between two quantities, each changing with respect to time, can aid in problem solving.

In a related rates problem, we attempt to find the rate of change of one quantity in terms of the rate of change of a second, related quantity that is, in some cases, easier to calculate.

In This Module

- We will explore different related rates problems.

Introducing Related Rates

Consider a circle whose radius is changing with respect to time.

As the radius changes, so does the area of the circle.

Let $r(t)$ be the radius of the circle at time t and let $A(t)$ be the area of the circle at time t.

Of course, we know that these two quantities are not independent.

More precisely, they are connected by the formula $A = \pi r^2$.

Examples

Example 1

If the radius of a circle is increasing at a constant rate of 2 cm/s, at what rate is the area of the circle changing when the radius is 3 cm?

Solution

Since we know the relationship between the quantities A and r, we can find the relationship between the rates of change of these quantities with respect to time.
We do so by differentiating both sides of the equation $A = \pi r^2$, implicitly, with respect to time.
Remember that, although we do not explicitly include the variable t in the equation, A and r are functions of time and π is a constant.

Examples

Example 1

If the radius of a circle is increasing at a constant rate of 2 cm/s, at what rate is the area of the circle changing when the radius is 3 cm?

Solution

$$A = \pi r^2$$

We are given that $\frac{dr}{dt} = 2$ cm/s and we are asked to determine the value of $\frac{dA}{dt}$.

$$\begin{aligned}\frac{dA}{dt} &= \frac{d}{dt}\left(\pi r^2\right)\\ &= \pi \frac{d}{dt}\left(r^2\right)\\ &= \pi(2r)\left(\frac{dr}{dt}\right) && \text{by the chain rule}\\ &= 2\pi r\left(\frac{dr}{dt}\right)\end{aligned}$$

Right away we see that even though the radius is changing at a constant rate, the area is not.
The rate of change of the area depends on both $\frac{dr}{dt}$ (which is constant) and r (which is changing with respect to time).

We want the rate of change of the area of the circle at the instant when the radius is 3 cm.

Therefore, we want to calculate $\frac{dA}{dt}$ when $r = 3$.

Examples

Example 1

If the radius of a circle is increasing at a constant rate of 2 cm/s, at what rate is the area of the circle changing when the radius is 3 cm?

Solution

Recall that $\frac{dr}{dt} = 2$ cm/s. Substituting $\frac{dr}{dt} = 2$ cm/s and $r = 3$ cm into the above equation, we get

$$\frac{dA}{dt} = 2\pi r\left(\frac{dr}{dt}\right) = 2\pi(3)(2) = 12\pi$$

Checking our units for $\frac{dA}{dt}$, we get (cm)×(cm/s)=cm^2/s as we would expect.

Therefore, the area of the circle is **increasing** at a rate of 12π cm^2/s when the radius is 3 cm.

Examples

Example 2

Suppose that a circular puddle is evaporating due to the sun and that the radius of the puddle is decreasing at a constant rate of 1 mm/h. At what rate is the circumference of the puddle changing when the circumference is 50 mm?

Solution

The related quantities in this question are the radius, r, and circumference, C, of the circular puddle.

We are given that $\frac{dr}{dt} = -1$ mm/h. (There is a negative sign as the radius is decreasing.)

We are asked to find the value of $\frac{dC}{dt}$ when $C = 50$ mm.

We know that the quantities r and C are related by the equation $C = 2\pi r$.

Now, we implicitly differentiate the equation for the circumference with respect to time:

$$\begin{aligned}\frac{dC}{dt} &= \frac{d}{dt}\left(2\pi r\right)\\ &= \frac{d}{dr}\left(2\pi r\right)\left(\frac{dr}{dt}\right) && \text{by the chain rule}\\ &= 2\pi\left(\frac{dr}{dt}\right)\end{aligned}$$

Examples

Example 2

Suppose that a circular puddle is evaporating due to the sun and that the radius of the puddle is decreasing at a constant rate of 1 mm/h. At what rate is the circumference of the puddle changing when the circumference is 50 mm?

Solution

$$\frac{dC}{dt} = 2\pi\left(\frac{dr}{dt}\right)$$

Here we see that $\frac{dC}{dt}$ depends only on the quantity $\frac{dr}{dt}$ which is constant. Hence, $\frac{dC}{dt}$ is also constant.
Substituting the value of $\frac{dr}{dt} = -1$, we get

$$\frac{dC}{dt} = 2\pi(-1) = -2\pi$$

and so the circumference is **decreasing** at a rate of 2π mm/h when $C = 50$ mm (and at any other time).

Remarks

Most related rates problems involve changes in geometric or physical quantities (length, speed, distance, etc).

We have seen one problem involving two related one-dimensional quantities (lengths) and a problem involving the relationship between a one-dimensional quantity (length) and two-dimensional quantity (area).
The same process applies when dealing with three-dimensional quantities.

Examples

Example 3

Suppose that a spherical balloon is being inflated at a rate of 5 cm^3/s. At what rate is the radius of the balloon changing when the radius is 9 cm?
At what rate is the radius of the balloon changing when the volume is 288π cm^3?

Solution

The balloon is a sphere that is being inflated, so its volume is changing and, of course, the radius would need to change as well. The related quantities are r, the radius, and V, the volume, of the spherical balloon.
For the first part of the question, we have the following.

Given: $\frac{dV}{dt} = 5$ cm^3/s.   Find: $\frac{dr}{dt}$ when $r = 9$ cm.

Relationship: $V = \frac{4}{3}\pi r^3$ (the volume of a sphere with radius r).

$$\begin{aligned}\frac{dV}{dt} &= \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) \\ &= \frac{4}{3}\pi(3r^2)\left(\frac{dr}{dt}\right) && \text{by the chain rule} \\ &= 4\pi r^2\left(\frac{dr}{dt}\right)\end{aligned}$$

Examples

Example 3

Suppose that a spherical balloon is being inflated at a rate of 5 cm^3/s. At what rate is the radius of the balloon changing when the radius is 9 cm?
At what rate is the radius of the balloon changing when the volume is 288π cm^3?

Solution

$$\frac{dV}{dt} = 4\pi r^2 \left(\frac{dr}{dt}\right)$$

Substituting the known value of $\frac{dV}{dt} = 5$ and the radius of $r = 9$ into the related rates equation, we get

$$5 = 4\pi(9)^2 \left(\frac{dr}{dt}\right) = 324\pi \left(\frac{dr}{dt}\right)$$

Rearranging the equation, we get

$$\frac{dr}{dt} = \frac{5}{324\pi}$$

and so the radius is **increasing** at a rate of $\frac{5}{324\pi}$ cm/s when the radius is 9 cm.

Examples

Example 3

Suppose that a spherical balloon is being inflated at a rate of 5 cm^3/s. At what rate is the radius of the balloon changing when the radius is 9 cm?
At what rate is the radius of the balloon changing when the volume is 288π cm^3?

Solution

To answer the second part of the question, we can use the same related rates equation:

$$\frac{dV}{dt} = 4\pi r^2 \left(\frac{dr}{dt}\right)$$

As before, we are given $\frac{dV}{dt} = 5$, but now we want to find $\frac{dr}{dt}$ when $V = 288\pi$ cm^3.
To do so, we need to find the radius of the balloon when the volume is 288π cm^3.

Examples

Example 3

Suppose that a spherical balloon is being inflated at a rate of 5 cm^3/s. At what rate is the radius of the balloon changing when the radius is 9 cm?
At what rate is the radius of the balloon changing when the volume is 288π cm^3?

Solution

$$V = \frac{4}{3}\pi r^3 = 288\pi$$

$$r^3 = 288\left(\frac{3}{4}\right) = 216$$

$$r = 6$$

Substituting the values $\frac{dV}{dt} = 5$ and $r = 6$ into the related rates equation, we get

$$5 = 4\pi(6)^2\left(\frac{dr}{dt}\right) = 144\pi\left(\frac{dr}{dt}\right)$$

and so

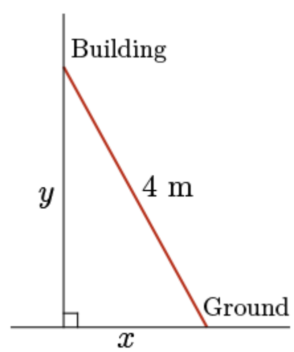
$$\frac{dr}{dt} = \frac{5}{144\pi}$$

Hence, the radius is **increasing** at a rate of $\frac{5}{144\pi}$ cm/s when the volume is 288π cm^3.

Examples

Example 4

A 4 m ladder rests against the side of a building. The base of the ladder begins to slip at a constant rate of 0.3 m/min. How fast is the top of the ladder sliding down the building when the base of the ladder is 2 m from the building?



Solution

First, we draw a diagram of the physical situation.
Let x represent the distance from the base of the ladder to the building (in metres).
Let y represent the distance from the top of the ladder to the ground (in metres).
The related quantities are x and y, both changing as a function of time.
Given: $\frac{dx}{dt} = 0.3$ m/min. Find: $\frac{dy}{dt}$ when $x = 2$ m.
Relationship: by the Pythagorean theorem, we have $x^2 + y^2 = 4^2 = 16$.
Using implicit differentiation, with respect to time, we get

$$\frac{d}{dt}\left(x^2 + y^2\right) = \frac{d}{dt}\left(x^2\right) + \frac{d}{dt}\left(y^2\right) = \frac{d}{dt}\left(16\right) \quad \text{by the sum rule for the derivative}$$

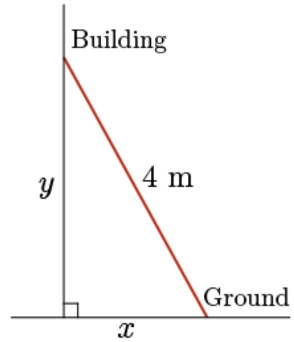
$$\frac{d}{dx}\left(x^2\right)\left(\frac{dx}{dt}\right) + \frac{d}{dy}\left(y^2\right)\left(\frac{dy}{dt}\right) = 0 \quad \text{by the chain rule}$$

$$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 0$$

Examples

Example 4

A 4 m ladder rests against the side of a building. The base of the ladder begins to slip at a constant rate of 0.3 m/min. How fast is the top of the ladder sliding down the building when the base of the ladder is 2 m from the building?



Solution

$$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 0$$

Solving for $\frac{dy}{dt}$ in this equation, we get

$$2y\left(\frac{dy}{dt}\right) = -2x\left(\frac{dx}{dt}\right)$$
$$\frac{dy}{dt} = -\frac{x}{y}\left(\frac{dx}{dt}\right), \;\; y \neq 0$$

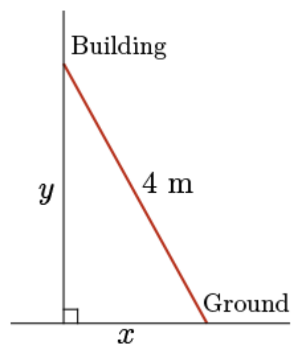
Note that if $\frac{dx}{dt} > 0$, then $\frac{dy}{dt} < 0$ and if $\frac{dx}{dt} < 0$, then $\frac{dy}{dt} > 0$. Why does this make sense in the physical situation?

Therefore, to find the value of $\frac{dy}{dt}$, we need to first find the value of y at the instant that $x = 2$.

Examples

Example 4

A 4 m ladder rests against the side of a building. The base of the ladder begins to slip at a constant rate of 0.3 m/min. How fast is the top of the ladder sliding down the building when the base of the ladder is 2 m from the building?



Solution

Using the Pythagorean theorem again, we get $y^2 = 16 - x^2 = 16 - 2^2 = 12$ and hence $y = \sqrt{12} = 2\sqrt{3}$ as $y > 0$.

Substituting the values for x, y, and $\frac{dx}{dt}$ into our equation gives

$$\frac{dy}{dt} = -\frac{x}{y}\left(\frac{dx}{dt}\right) = -\frac{2}{2\sqrt{3}}(0.3) = -\frac{0.3}{\sqrt{3}} \approx -0.17$$

This means that the ladder is sliding **down** the wall at a rate of approximately 0.17 m/min when $x = 2$.

Examples

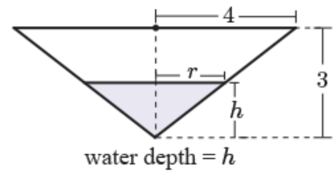
Challenge Question

A conical reservoir is filling with water at a constant rate of 3 m^3/min. The reservoir is 3 m deep and has a diameter of 8 m at the opening. Determine the rate at which the depth of the water is increasing when the depth of the water is 2 m.

Solution

The related quantities in this problem are the volume, V, of water in the reservoir and the depth, h, of the water.

First, we draw a diagram of the physical situation.



Given: $\dfrac{dV}{dt} = 3$ m^3/min. Find: $\dfrac{dh}{dt}$ when $h = 2$ m.

What is the relationship between V and h?

Recall that the volume of a cone is a function of both its height, h, and its radius, r. Therefore, there is a third quantity, r, that is important in this problem (although not explicitly mentioned).

Let r be the radius of the (circular) surface of the water, which also changes with respect to time.

Now the volume of water in the reservoir can be calculated in terms of h and r as follows:

Relationship: $V = \dfrac{1}{3}\pi r^2 h$ (volume of a cone with radius r and height h)

We could proceed, as usual, by differentiating the formula with respect to time, but since we have two quantities on the right hand side, both changing with respect to time, we would need to use the product rule.

Also, we would introduce the term $\dfrac{dr}{dt}$ when differentiating, and we are given no information about this rate of change in the question.

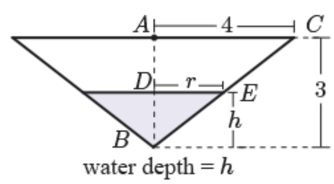
Examples

Challenge Question

A conical reservoir is filling with water at a constant rate of 3 m^3/min. The reservoir is 3 m deep and has a diameter of 8 m at the opening. Determine the rate at which the depth of the water is increasing when the depth of the water is 2 m.

Solution

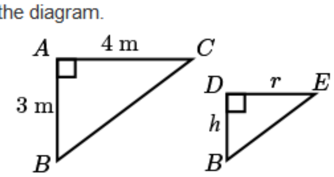
In this case, there is a simple formula relating the quantities h and r and we will use this to eliminate the quantity r.



Consider the following two triangles from the diagram.

Since $\angle A = 90^\circ = \angle D$ and $\angle B$ is common in each triangle, we have that ΔABC is similar to ΔDBE.

It follows that $\dfrac{3}{h} = \dfrac{4}{r} \implies r = \dfrac{4}{3}h$.



Examples

Challenge Question

A conical reservoir is filling with water at a constant rate of 3 m^3/min. The reservoir is 3 m deep and has a diameter of 8 m at the opening. Determine the rate at which the depth of the water is increasing when the depth of the water is 2 m.

Solution

Substituting $r = \frac{4}{3}h$ into our principal equation, we get

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi\left(\frac{4}{3}h\right)^2 h \\ &= \frac{16}{27}\pi h^3 \end{aligned}$$

Examples

Challenge Question

A conical reservoir is filling with water at a constant rate of 3 m^3/min. The reservoir is 3 m deep and has a diameter of 8 m at the opening. Determine the rate at which the depth of the water is increasing when the depth of the water is 2 m.

Solution

$$V = \frac{16}{27}\pi h^3$$

Differentiating the equation, implicitly, with respect to t gives

$$\begin{aligned} \frac{dV}{dt} &= \frac{16}{27}\pi\left(3h^2\right)\left(\frac{dh}{dt}\right) \\ \frac{dV}{dt} &= \frac{16}{9}\pi h^2\left(\frac{dh}{dt}\right) \\ 3 &= \frac{16}{9}\pi(2)^2\left(\frac{dh}{dt}\right) \qquad \left(\frac{dV}{dt} = 3,\ h = 2\right) \\ \frac{dh}{dt} &= \frac{27}{64\pi} \end{aligned}$$

Therefore, the depth of the water is increasing at a rate of $\frac{27}{64\pi}$ m/s when the depth is 2 m.

General Strategy for Related Rates Problems

- Determine the related quantities from the question $(V,\ A,\ C,\ h,\ r, \ldots)$ and identify the principal formula.

  Circle: $C = 2\pi r, \quad A = \pi r^2$

  Sphere: $V = \dfrac{4}{3}\pi r^3, \quad \text{Surface Area } = 4\pi r^2$

  Cone: $V = \dfrac{1}{3}\pi r^2 h$

  Cylinder: $V = \pi r^2 h, \quad \text{Surface Area } = 2\pi r^2 + 2\pi r h$

  Pythagorean Relationship: $x^2 + y^2 = c^2$

- Can you draw a diagram of the physical situation?

- What are you given? $\dfrac{dV}{dt}, \dfrac{dr}{dt}, \dfrac{dh}{dt}, \ldots$

- What are you asked to find? $\dfrac{dV}{dt}, \dfrac{dr}{dt}, \dfrac{dh}{dt}, \ldots$

- If there are more than two quantities (for example, V, h, and r in the cone problem), then it may be necessary to express one of the variables in terms of another (using a method like similar triangles) and then substitute back into the primary equation, thus, eliminating one quantity.

- **Never** substitute a value for a variable until **after** you have differentiated the principal formula.