The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

The Dot Product of Two Vectors

Recall

So far, addition, subtraction, and scalar multiplication of vectors have been introduced and explored.
While each of these operations has an intuitive geometrical interpretation, they alone are unable to handle certain applications of vectors.

In this Module

- We will develop two new operations that will allow us to extend our ability to apply vectors to physical and geometrical situations.
- We will examine a new vector operation: the dot product.

Definition

The **dot product** of two vectors

$$\underbrace{\vec{u}\cdot\vec{v}}_{u \text{ dot } v} = |\vec{u}||\vec{v}|\cos(\theta)$$

where θ is the angle between the vectors $\vec{u}$ and $\vec{v}$.

Examples

Example 1

Find the dot product of $\vec{u}$ and $\vec{v}$ given that $|\vec{u}| = 8, |\vec{v}| = 3$, and the angle between $\vec{u}$ and $\vec{v}$ is

a. $\theta = 40°$

b. $\theta = \frac{3\pi}{4}$

Solution

a.

$$\begin{aligned}\vec{u}\cdot\vec{v} &= |\vec{u}||\vec{v}|\cos(\theta)\\ &= (8)(3)\cos(40°)\\ &\approx 18.385\end{aligned}$$

b.

$$\begin{aligned}\vec{u}\cdot\vec{v} &= |\vec{u}||\vec{v}|\cos(\theta)\\ &= (8)(3)\cos\left(\frac{3\pi}{4}\right)\\ &= 24\left(-\frac{1}{\sqrt{2}}\right)\\ &= -12\sqrt{2}\end{aligned}$$

Note:

Since $|\vec{u}|$, $|\vec{v}|$, and $\cos(\theta)$ are all scalars (real numbers), their product will be a scalar.
Thus, unlike the previously learned vector operations (which produced vectors as answers), the dot product always yields a scalar answer.
For this reason, dot product is often referred to as the scalar product.

Examples

Example 2

Prove that if the dot product of two non-zero vectors, $\vec{u}$ and $\vec{v}$, is equal to zero, then $\vec{u}$ and $\vec{v}$ must be perpendicular.

Solution

We are required to prove $\vec{u} \perp \vec{v}$ given that $\vec{u} \cdot \vec{v} = 0$.

Proof

$$\begin{aligned} \vec{u} \cdot \vec{v} &= 0 \\ \therefore |\vec{u}||\vec{v}| \cos(\theta) &= 0 \end{aligned}$$

Since $|\vec{u}| \neq 0$ and $|\vec{v}| \neq 0$, it must be that $\cos(\theta) = 0$.
Solving this equation, we get $\theta = \pm 90^\circ$.
Therefore, $\vec{u}$ is perpendicular to $\vec{v}$.

Is the converse true? That is, if non-zero vectors, $\vec{u}$ and $\vec{v}$, are perpendicular, then does $\vec{u} \cdot \vec{v} = 0$?
This example proves an important property of the dot product of two vectors:

> If $\vec{u} \cdot \vec{v} = 0$ where $\vec{u}, \vec{v} \neq \vec{0}$, then $\vec{u} \perp \vec{v}$. Conversely, if $\vec{u} \perp \vec{v}$, then $\vec{u} \cdot \vec{v} = 0$.

Two vectors are **orthogonal** if they are perpendicular, but includes the case where either (possibly both) vectors are $\vec{0}$.

Properties of Dot Product

1. $a(\vec{u} \cdot \vec{v}) = (a\vec{u}) \cdot \vec{v} = \vec{u} \cdot (a\vec{v})$
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$

Proof of 3

$$\begin{aligned} \vec{u} \cdot \vec{u} &= |\vec{u}||\vec{u}| \cos(\theta) \\ \text{since } \theta &= 0^\circ \\ \cos(\theta) &= 1 \\ \therefore \vec{u} \cdot \vec{u} &= |\vec{u}|^2 \end{aligned}$$

Examples

Example 3

Evaluate each of the following dot products:

a. $\hat{i} \cdot \hat{i}$

b. $\hat{i} \cdot \hat{j}$

Solution

a.

$$\begin{aligned}\hat{i} \cdot \hat{i} &= |\hat{i}|^2 \\ &= (1)^2 \\ &= 1\end{aligned}$$

b.

$$\hat{i} \cdot \hat{j} = 0$$

since $\hat{i} \perp \hat{j}$

Examples

Example 4

Given $\vec{a} = (a_x, a_y)$ and $\vec{b} = (b_x, b_y)$, prove that $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y$.

Solution

$$\begin{aligned}\vec{a} &= (a_x, a_y) \\ &= (a_x, 0) + (0, a_y) \\ &= a_x(1, 0) + a_y(0, 1) \\ &= a_x\hat{i} + a_y\hat{j}\end{aligned}$$

Similarly, $\vec{b} = b_x\hat{i} + b_y\hat{j}$.

$$\begin{aligned}\therefore \vec{a} \cdot \vec{b} &= (a_x\hat{i} + a_y\hat{j}) \cdot (b_x\hat{i} + b_y\hat{j}) \\ &= a_x b_x(\hat{i} \cdot \hat{i}) + a_x b_y(\hat{i} \cdot \hat{j}) + a_y b_x(\hat{j} \cdot \hat{i}) + a_y b_y(\hat{j} \cdot \hat{j}) \\ &= a_x b_x(1) + a_x b_y(0) + a_y b_x(0) + a_y b_y(1) \\ &= a_x b_x + a_y b_y\end{aligned}$$

This property extends to 3-dimensional algebraic vectors. That is, if $\vec{a} = (a_x, a_y, a_z)$ and $\vec{b} = (b_x, b_y, b_z)$, then

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

Examples

Example 5

Given $\vec{a}$ and $\vec{b}$, determine their dot product.

a. $\vec{a} = (2, -1)$ and $\vec{b} = (4, 3)$

b. $\vec{a} = (1, 0, 3)$ and $\vec{b} = (-2, 5, 8)$

Solution

a.

$$\begin{aligned}\vec{a} \cdot \vec{b} &= (2, -1) \cdot (4, 3) \\ &= (2)(4) + (-1)(3) \\ &= 5\end{aligned}$$

b.

$$\begin{aligned}\vec{a} \cdot \vec{b} &= (1, 0, 3) \cdot (-2, 5, 8) \\ &= (1)(-2) + (0)(5) + (3)(8) \\ &= 22\end{aligned}$$

For example, in 3-dimensional space, given $\vec{u} = (u_x, u_y, u_z)$ and $\vec{v} = (v_x, v_y, v_z)$, we saw that $\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z v_z$ and we also saw that $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos(\theta)$.

We have now seen two definitions of dot product.

By comparing these two definitions, we see that

$$\begin{aligned}\vec{u} \cdot \vec{v} &= |\vec{u}||\vec{v}| \cos(\theta) \\ \therefore u_x v_x + u_y v_y + u_z v_z &= |\vec{u}||\vec{v}| \cos(\theta)\end{aligned}$$

Aside

A similar definition holds for vectors with dimensions greater than three.

It is possible, and often useful, to write vectors with dimensions greater than three.

For example, consider the question: How many dimensions does it take to specify the exact position of an airplane in 3 space?

We need

- three components to describe the centre position of the airplane (x, y, z);
- a component for the pitch (up/down rotation);
- a component for yaw (left/right rotation); and
- a component for the roll (nose-tail axis rotation).

Thus, the exact position of a plane can be represented by a vector with six components.

Examples

Example 6

Find the angle θ between each of the following pairs of vectors.

a. $\vec{u} = (2, -1)$ and $\vec{v} = (3, -4)$

b. $\vec{u} = (2, 5, -1)$ and $\vec{v} = (17, -7, -1)$

Solution

a.

$$\begin{aligned}
\vec{u} \cdot \vec{v} &= |\vec{u}||\vec{v}| \cos(\theta) \\
u_x v_x + u_y v_y &= |\vec{u}||\vec{v}| \cos(\theta) \\
6 + 4 &= \sqrt{5}\sqrt{25} \cos(\theta) \\
\frac{10}{5\sqrt{5}} &= \cos(\theta) \\
\frac{2}{\sqrt{5}} &= \cos(\theta) \\
\theta &\approx 27^\circ \quad (\text{since } 0^\circ \le \theta \le 180^\circ)
\end{aligned}$$

b.

$$\begin{aligned}
\vec{u} \cdot \vec{v} &= |\vec{u}||\vec{v}| \cos(\theta) \\
u_x v_x + u_y v_y + u_z v_z &= |\vec{u}||\vec{v}| \cos(\theta) \\
34 - 35 + 1 &= |\vec{u}||\vec{v}| \cos(\theta) \\
0 &= \cos(\theta) \\
\theta &= 90^\circ \quad (\text{since } 0^\circ \le \theta \le 180^\circ)
\end{aligned}$$

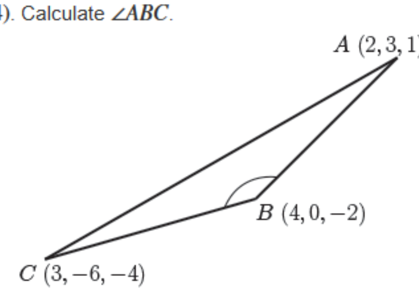
Examples

Example 7

Triangle ABC has vertices $A\ (2, 3, 1)$, $B\ (4, 0, -2)$, and $C\ (3, -6, -4)$. Calculate $\angle ABC$.

Solution

$$\begin{aligned}
\overrightarrow{BA} &= (-2, 3, 3) \\
\overrightarrow{BC} &= (-1, -6, -2) \\
\overrightarrow{BA} \cdot \overrightarrow{BC} &= |\overrightarrow{BA}||\overrightarrow{BC}| \cos(\angle ABC) \\
-\frac{22}{\sqrt{22}\sqrt{41}} &= \cos(\angle ABC) \\
\angle ABC &\approx 137^\circ
\end{aligned}$$



Examples

Example 8

Calculate the exact value of $(4\vec{x} - \vec{y}) \cdot (2\vec{x} + 3\vec{y})$ if $|\vec{x}| = 3$, $|\vec{y}| = 4$, and the angle between $\vec{x}$ and $\vec{y}$ is 150°.

Solution

$$\begin{aligned}
(4\vec{x} - \vec{y}) \cdot (2\vec{x} + 3\vec{y}) &= 8|\vec{x}|^2 + 12(\vec{x} \cdot \vec{y}) - 2(\vec{x} \cdot \vec{y}) - 3|\vec{y}|^2 \\
&= 8(3)^2 + 10(\vec{x} \cdot \vec{y}) - 3(4)^2 \\
&= 72 + 10\Big((3)(4)\cos(150^\circ)\Big) - 48 \\
&= 24 + 10\left(12\left(-\frac{\sqrt{3}}{2}\right)\right) \\
&= 24 + 10(-6\sqrt{3}) \\
&= 24 - 60\sqrt{3}
\end{aligned}$$

Examples

Example 9

The magnitude of the sum of vectors $\vec{a}$ and $\vec{b}$ is equal to the magnitude of their difference. Determine the angle between $\vec{a}$ and $\vec{b}$.

Solution

We are given

$$\begin{gathered}
|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}| \\
|\vec{a} + \vec{b}|^2 = |\vec{a} - \vec{b}|^2 \\
(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \\
\vec{a} \cdot \vec{a} + 2(\vec{a} \cdot \vec{b}) + \vec{b} \cdot \vec{b} = \vec{a} \cdot \vec{a} - 2(\vec{a} \cdot \vec{b}) + \vec{b} \cdot \vec{b} \\
4(\vec{a} \cdot \vec{b}) = 0
\end{gathered}$$

Therefore, $\vec{a} \cdot \vec{b} = 0$ and so $\vec{a}$ and $\vec{b}$ are orthogonal vectors. If we assume they are both non-zero vectors, then we may say that they are perpendicular.

The angle between non-zero vectors $\vec{a}$ and $\vec{b}$ is 90°.