The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Equations Of Lines In R3

Introduction

Three ways of describing a line in $\mathbb{R}^2$ have been discussed so far:

Vector Equation

$$\vec{r} = \vec{r}_0 + t\vec{d}, t \in \mathbb{R}$$

Parametric Equations

$$x = x_0 + ta,$$
$$y = y_0 + tb,\ t \in \mathbb{R}$$

Scalar Equation

$$Ax + By + C = 0$$

There is a natural extension of the vector and parametric equation descriptions to lines in $\mathbb{R}^3$. However, the scalar equation does not generalize as it is defined by the normal vector to a line. Since there are infinitely many normals to a given line in 3 dimensions, there is no valid definition.

Vector Equation of a Line in $\mathbb{R}^3$

Similar to the case in 2 dimensions, the vector equation of a line in $\mathbb{R}^3$ can be described using a point on the line and a direction vector for the line.

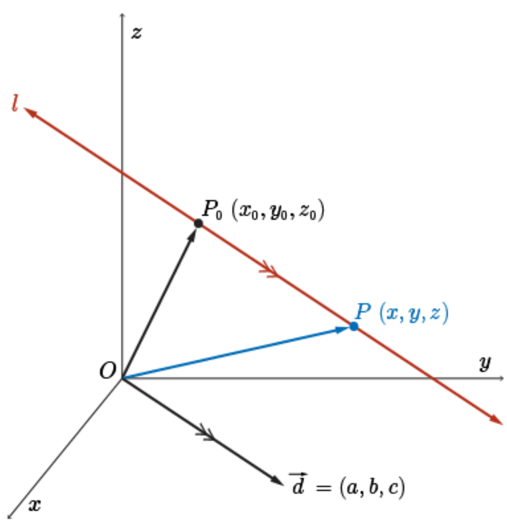
Let $P_0\ (x_0, y_0, z_0)$ be a specific point on the line.
Let $P\ (x, y, z)$ be an arbitrary point on the line.

Using the triangle law of vector addition,

$$\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$$
$$(x, y, z) = (x_0, y_0, z_0) + t\vec{d}$$
$$\text{or } \vec{r} = \vec{r}_0 + t\vec{d},\ t \in \mathbb{R}$$

where $\vec{d} = (a, b, c)$ is a direction vector for the line.

This is the vector equation of a line in $\mathbb{R}^3$.



Parametric Equations of a Line in $\mathbb{R}^3$

Considering the individual components of the vector equation of a line in 3-space gives the parametric equations

$$
\begin{aligned}
x &= x_0 + ta \\
y &= y_0 + tb \\
z &= z_0 + tc
\end{aligned}
$$

where $t \in \mathbb{R}$ and $\vec{d} = (a, b, c)$ is a direction vector of the line.

Using the three parametric equations and rearranging each to solve for t, gives the symmetric equations of a line in $\mathbb{R}^3$.

$$
t = \frac{x - x_0}{a}, \quad t = \frac{y - y_0}{b}, \quad t = \frac{z - z_0}{c}
$$
$$
\therefore \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}, \quad a \neq 0, b \neq 0, c \neq 0
$$

Examples

Example 1

a. Find the vector, parametric, and symmetric equations of the line through $P_0\ (3, 7, -2)$ with direction vector $(1, -3, 2)$.
b. Find two other points on the line.
c. Is $(-1, 19, 8)$ on the line?

Solution

a. The vector equation of the line is $(x, y, z) = (3, 7, -2) + t(1, -3, 2),\ t \in \mathbb{R}$.

The parametric equations of the line are

$$
\begin{aligned}
x &= 3 + t \\
y &= 7 - 3t \\
z &= -2 + 2t,\ t \in \mathbb{R}
\end{aligned}
$$

The symmetric equations are

$$
\frac{x - 3}{1} = \frac{y - 7}{-3} = \frac{z + 2}{2}
$$

b. Using the vector equation of the line, set $t = -1$ and $t = 1$ to find two points on the line.

When $t = -1$, we get $(x, y, z) = (3, 7, -2) - (1, -3, 2) = (2, 10, -4)$.
When $t = 1$, we get $(x, y, z) = (3, 7, -2) + (1, -3, 2) = (4, 4, 0)$.

Examples

Example 1

a. Find the vector, parametric, and symmetric equations of the line through $P_0\,(3,7,-2)$ with direction vector $(1,-3,2)$.
b. Find two other points on the line.
c. Is $(-1,19,8)$ on the line?

Solution

c. A method to determine if $(-1,19,8)$ is on the line is to solve for t using one of the parametric equations and substitute this value for t into the other two equations.
Using $x=3+t$, we get $-1=3+t$ and so $t=-4$.
Substituting $t=-4$ into the equation for y gives $y=7-3(-4)=19$, which is consistent with the point.
Substituting $t=-4$ into the equation for z gives $z=-2+2(-4)=-10$, which is not consistent with the point ($z=8$).
Therefore, the point $(-1,19,8)$ is not on the line.

An alternative method is to substitute the coordinates of the point into the symmetric equations and verify that the equations are consistent. Substituting gives

$$\frac{x-3}{1}=\frac{-1-3}{1}=-4 \qquad \frac{y-7}{-3}=\frac{19-7}{-3}=-4 \qquad \frac{z+2}{2}=\frac{8+2}{2}=5$$

which is inconsistent (since $-4\neq 5$), so the point $(-1,19,8)$ is not on the line.

Examples

Example 2

Find the parametric and symmetric equations of the line passing through $P\,(2,-5,3)$ and $Q\,(-4,-5,7)$.

Solution

Using the two points we may find a direction vector for the line; $\vec{d}=\overrightarrow{PQ}=(-6,0,4)$.
This gives the parametric equations

$$\begin{aligned} x&=2-6t\\ y&=-5\\ z&=3+4t,\ t\in\mathbb{R}\end{aligned}$$

The symmetric equation is

$$\frac{x-2}{-6}=\frac{z-3}{4}\,, y=-5$$

Since y is independent of the value of t, we write $y=-5$ as part of the symmetric equation.
This happens when we have a component of the direction vector equal to 0.

Further, if two of the components of a direction vector are equal to 0, then there are no symmetric equations for the line.

Examples

Example 3

Lines L_1 and L_2 are two lines in $\mathbb{R}^3$ defined as

$$L_1 = \overrightarrow{OP_1} + t\vec{d_1},\ t \in \mathbb{R}, \quad L_2 = \overrightarrow{OP_2} + s\vec{d_2},\ s \in \mathbb{R}$$

Prove that the lines are coincident **if and only if** $\overrightarrow{OP_1} - \overrightarrow{OP_2}$ and $\vec{d_1}$ are scalar multiples of $\vec{d_2}$.

Solution

First, we must discuss the phrase **if and only if**.
When asked to prove an "A if and only if B" statement, we are required to prove that A implies B (if A is true, then B is true), but also that B implies A (if B is true, then A is true).

In the example given, this means that we are required to prove the following two parts:

Part 1: Prove that if the lines are coincident (if $L_1 = L_2$), then $\overrightarrow{OP_1} - \overrightarrow{OP_2}$ and $\vec{d_1}$ are scalar multiples of $\vec{d_2}$, and the converse statement,

Part 2: Prove that if $\overrightarrow{OP_1} - \overrightarrow{OP_2}$ and $\vec{d_1}$ are scalar multiples of $\vec{d_2}$, then the lines are coincident.

Examples

Example 3

Part 1: Prove that if the lines are coincident (if $L_1 = L_2$), then $\overrightarrow{OP_1} - \overrightarrow{OP_2}$ and $\vec{d_1}$ are scalar multiples of $\vec{d_2}$.

Solution

Proof of Part 1

Prove that if the lines are coincident (if $L_1 = L_2$), then $\overrightarrow{OP_1} - \overrightarrow{OP_2}$ and $\vec{d_1}$ are scalar multiples of $\vec{d_2}$.
Part 1 has two statements to prove.
These are

i. if the lines are coincident (if $L_1 = L_2$), then $\overrightarrow{OP_1} - \overrightarrow{OP_2}$ is a scalar multiple of $\vec{d_2}$; and
ii. if the lines are coincident (if $L_1 = L_2$), then $\vec{d_1}$ is a scalar multiple of $\vec{d_2}$.

Proof of Part 1 (i)

Assume that $L_1 = L_2$.
We must prove that $\overrightarrow{OP_1} - \overrightarrow{OP_2}$ is a scalar multiple of $\vec{d_2}$.
Since $\overrightarrow{OP_1} \in L_1$ $\left(L_1 = \overrightarrow{OP_1} + t\vec{d_1}\right)$ and $L_1 = L_2$, then $\overrightarrow{OP_1} \in L_2$ $\left(L_2 = \overrightarrow{OP_2} + s\vec{d_2}\right)$.
Therefore, $\overrightarrow{OP_1} = \overrightarrow{OP_2} + s\vec{d_2}$ for some s (by the definition of L_2).
Then $\overrightarrow{OP_1} - \overrightarrow{OP_2} = s\vec{d_2}$, so $\overrightarrow{OP_1} - \overrightarrow{OP_2}$ is a scalar multiple of $\vec{d_2}$.

Examples

Example 3

Part 1: Prove that if the lines are coincident (if $L_1 = L_2$), then $\overrightarrow{OP_1} - \overrightarrow{OP_2}$ and $\vec{d_1}$ are scalar multiples of $\vec{d_2}$.

Solution

These are

i. if the lines are coincident (if $L_1 = L_2$), then $\overrightarrow{OP_1} - \overrightarrow{OP_2}$ is a scalar multiple of $\vec{d_2}$; and

ii. if the lines are coincident (if $L_1 = L_2$), then $\vec{d_1}$ is a scalar multiple of $\vec{d_2}$.

Proof of Part 1 (ii)

Assume that $L_1 = L_2$. We must prove that $\vec{d_1}$ is a scalar multiple of $\vec{d_2}$.

We have that $\overrightarrow{OP_1} + \vec{d_1} \in L_1$ (let $t = 1$ in $L_1 = \overrightarrow{OP_1} + t\vec{d_1}$) and since $L_1 = L_2$, then $\overrightarrow{OP_1} + \vec{d_1} \in L_2$.

Hence, we get $\overrightarrow{OP_1} + \vec{d_1} = \overrightarrow{OP_2} + s\vec{d_2}$ for some s (since $L_2 = \overrightarrow{OP_2} + s\vec{d_2}$).

Since $\overrightarrow{OP_1} \in L_1$ and $L_1 = L_2$, then $\overrightarrow{OP_1} \in L_2$ and so $\overrightarrow{OP_1} = \overrightarrow{OP_2} + q\vec{d_2}$ for some q.

Subtracting these two equations, $\overrightarrow{OP_1} + \vec{d_1} = \overrightarrow{OP_2} + s\vec{d_2}$ and $\overrightarrow{OP_1} = \overrightarrow{OP_2} + q\vec{d_2}$, we get $\vec{d_1} = s\vec{d_2} - q\vec{d_2}$ or $\vec{d_1} = (s - q)\vec{d_2}$ for some scalar $s - q$.

Hence, $\vec{d_1}$ is a scalar multiple of $\vec{d_2}$.

Therefore, if the lines are coincident, then $\overrightarrow{OP_1} - \overrightarrow{OP_2}$ and $\vec{d_1}$ are scalar multiples of $\vec{d_2}$.

Examples

Example 3

Part 2: Prove that if $\overrightarrow{OP_1} - \overrightarrow{OP_2}$ and $\vec{d_1}$ are scalar multiples of $\vec{d_2}$, then the lines are coincident.

Solution

Proof of Part 2

Prove that if $\overrightarrow{OP_1} - \overrightarrow{OP_2}$ and $\vec{d_1}$ are scalar multiples of $\vec{d_2}$, then the lines are coincident.

The line L_1 is defined by vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_1} + \vec{d_1}$.

If we can show that L_2 is also defined by these same two vectors, $\overrightarrow{OP_1}$ and $\overrightarrow{OP_1} + \vec{d_1}$, then $L_1 = L_2$.

Since $\overrightarrow{OP_1} - \overrightarrow{OP_2}$ is a scalar multiple of $\vec{d_2}$, we have $\overrightarrow{OP_1} - \overrightarrow{OP_2} = c\vec{d_2}$ or $\overrightarrow{OP_1} = \overrightarrow{OP_2} + c\vec{d_2}$ for some c.

Since $\overrightarrow{OP_1} = \overrightarrow{OP_2} + c\vec{d_2}$, then $\overrightarrow{OP_1} \in L_2$.

Similarly, $\vec{d_1}$ is a scalar multiple of $\vec{d_2}$ and so $\vec{d_1} = k\vec{d_2}$ for some k.

Examples

Example 3

Part 2: Prove that if $\overrightarrow{OP_1} - \overrightarrow{OP_2}$ and $\overrightarrow{d_1}$ are scalar multiples of $\overrightarrow{d_2}$, then the lines are coincident.

Solution

Proof of Part 2

Adding this equation to our previous equation $\overrightarrow{OP_1} = \overrightarrow{OP_2} + c\overrightarrow{d_2}$, we get

$$\begin{aligned}\overrightarrow{OP_1} + \overrightarrow{d_1} &= \overrightarrow{OP_2} + c\overrightarrow{d_2} + k\overrightarrow{d_2} \\ &= \overrightarrow{OP_2} + (c+k)\overrightarrow{d_2} \\ &= \overrightarrow{OP_2} + l\overrightarrow{d_2}, \text{ for some } l\end{aligned}$$

Therefore, $\overrightarrow{OP_1} + \overrightarrow{d_1} \in L_2$.

Thus, L_2 is defined by the same two vectors, $\overrightarrow{OP_1}$ and $\overrightarrow{OP_1} + \overrightarrow{d_1}$, as L_1 and so $L_1 = L_2$.

The lines L_1 and L_2 are coincident *if and only if* $\overrightarrow{OP_1} - \overrightarrow{OP_2}$ and $\overrightarrow{d_1}$ are scalar multiples of $\overrightarrow{d_2}$.

□

Examples

Example 4

Determine if the lines $L_1 = (-1, 0, 2) + t(2, -6, -2),\ t \in \mathbb{R}$ and $L_2 = (-22, 63, 23) + s(-7, 21, 7),\ s \in \mathbb{R}$ are coincident.

Solution

From the previous example, we know that the two lines $L_1 = \overrightarrow{OP_1} + t\overrightarrow{d_1},\ t \in \mathbb{R}$ and $L_2 = \overrightarrow{OP_2} + s\overrightarrow{d_2},\ s \in \mathbb{R}$ are coincident if $\overrightarrow{OP_1} - \overrightarrow{OP_2}$ and $\overrightarrow{d_1}$ are scalar multiples of $\overrightarrow{d_2}$.

In our example, if $(-1, 0, 2) - (-22, 63, 23)$ and $(2, -6, -2)$ are scalar multiples of $(-7, 21, 7)$, then L_1 and L_2 are coincident.

Clearly, $(2, -6, -2) = -\frac{2}{7}(-7, 21, 7)$ and so $(2, -6, -2)$ is a scalar multiple of $(-7, 21, 7)$.
Since $(-1, 0, 2) - (-22, 63, 23) = (21, -63, -21) = -3(-7, 21, 7)$, then $(-1, 0, 2) - (-22, 63, 23)$ is a scalar multiple of $(-7, 21, 7)$.

Therefore, the lines L_1 and L_2 are coincident.