Properties and Laws of Logarithms

Introduction

When working with exponents, we employ a variety of properties and laws to help simplify and evaluate exponential expressions.

Properties	Laws/Rules
$c^0 = 1,\ c \neq 0$	Product of Powers: $(c^m)(c^n) = c^{m+n}$
$c^{-n} = \dfrac{1}{c^n}$	Quotient of Powers: $\dfrac{c^m}{c^n} = c^{m-n}$
$c^{\frac{1}{n}} = \sqrt[n]{c}$	A Power of a Power: $(c^m)^n = c^{mn}$

Introduction

Common Logarithms

A logarithm base 10 is called a **common logarithm**. For simplicity, $\log_{10}(x)$ is often written $\log(x)$, with base 10 understood.

Common logarithms were the first logarithms introduced to carry out complicated calculations in the decimal number system (base 10).
The log function [log] on your calculator works in base 10.

Examples

Example 1

Evaluate.

a. $\log(0.001)$
b. $\log(70)$
c. $\log(-100)$

Solution

a. Evaluating this logarithm, we have

$$\begin{aligned}\log_{10}(0.001) &= \log_{10}\left(\frac{1}{1000}\right)\\ &= \log_{10}(10^{-3})\\ &= -3\end{aligned}$$

You can also use the log function on your calculator [log]. This works in base 10 (common base).

b. $\log(70) = 1.84509804\ldots$

Since this answer is the value of an exponent, it is important to keep 3 or 4 decimal places in your answer when rounding.

Therefore, $\log(70) \approx 1.845$. This means that $10^{1.845} \approx 70$.

Examples

Example 1

Evaluate.

a. $\log(0.001)$
b. $\log(70)$
c. $\log(-100)$

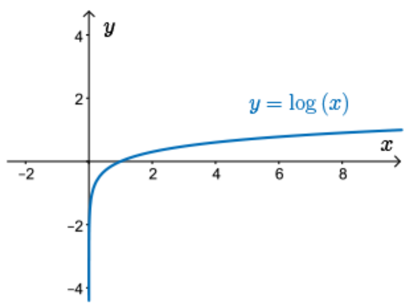
Solution

c.The calculator returns an error when asked to determine $\log(-100)$.

Why is this?

If we let $\log(-100) = n$, then $10^n = -100$.
There is no value for n such that 10^n produces a negative value since the base is positive.

Also, remember that the domain of the logarithmic function $y = \log_c(x), c > 0,\ c \neq 1$ is $\{x \mid x > 0, x \in \mathbb{R}\}$.



Examples

Example 2

Evaluate.

a. $4^{\log_4(64)}$ **b.** $5^{\log_5\left(\frac{1}{25}\right)}$ **c.** $3^{\log_3(20)}$

Solution

a. We have

$$\begin{aligned} 4^{\log_4(64)} &= 4^3 \\ &= 64 \end{aligned}$$

b. We have

$$\begin{aligned} 5^{\log_5\left(\frac{1}{25}\right)} &= 5^{-2} \\ &= \frac{1}{25} \end{aligned}$$

Notice $4^{\log_4(64)} = 64$ and $5^{\log_5\left(\frac{1}{25}\right)} = \frac{1}{25}$. It would seem that $3^{\log_3(20)}$ should equal 20.

c. If we let $3^{\log_3(20)} = n$ and convert to logarithmic form, we have

$$\begin{aligned} \log_3(n) &= \log_3(20) \\ \therefore n &= 20 \end{aligned}$$

Thus, $3^{\log_3(20)} = 20$.

> In general, $c^{\log_c(n)} = n$ for all $c > 0,\ c \neq 1,\ n > 0$.
> Similarly, $\log_c(c^n) = n$ for all $c > 0,\ c \neq 1,\ n \in \mathbb{R}$.

Examples

Example 3

Solve for x.

a. $\log_{16}(\sqrt{8}) = x$ **b.** $\log_x(15) = 2$ **c.** $\log_4(x) = 0$

Solve each of these equations using the equivalent exponential form:

$$\log_c(m) = n \iff m = c^n \text{ when } c > 0,\ c \neq 1,\ m > 0$$

Solution

a. $\log_{16}(\sqrt{8}) = x \implies$

$$\begin{aligned} 16^x &= 8^{\frac{1}{2}} \\ (2^4)^x &= (2^3)^{\frac{1}{2}} \\ 4x &= \frac{3}{2} \\ \therefore x &= \frac{3}{8} \end{aligned}$$

b. $\log_x(15) = 2 \implies$

$$\begin{aligned} x^2 &= 15 \\ x &= \pm\sqrt{15} \text{ but } x > 0 \\ \therefore x &= \sqrt{15} \end{aligned}$$

c. $\log_4(x) = 0 \implies$

$$\begin{aligned} 4^0 &= x \\ \therefore x &= 1 \end{aligned}$$

Note:

> $\log_c(1) = 0$, for all $c > 0$, $c \neq 1$. This follows from the exponential property $c^0 = 1$, $c \neq 0$.

Examples

Example 4

Solve $\log_9(\log_8(x-3)) = -\dfrac{1}{2}$.

Solution

First note the restrictions on x.

$$\begin{aligned} x-3 &> 0 \\ x &> 3 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \log_8(x-3) &> 0 \\ x-3 &> 1 \\ x &> 4 \end{aligned}$$

Thus, $x > 4$.

Most logarithmic equations can be solved using the equivalent exponential form. We begin with the conversion

$$\log_9(\log_8(x-3)) = -\frac{1}{2} \quad \Longrightarrow \quad 9^{-\frac{1}{2}} = \log_8(x-3)$$

so $\log_8(x-3) = \dfrac{1}{3}$.

We now convert this statement to exponential form.

$$\log_8(x-3) = \frac{1}{3} \quad \Longrightarrow \quad \begin{aligned} 8^{\frac{1}{3}} &= x-3 \\ 2 &= x-3 \\ x &= 5 \end{aligned}$$

Therefore, $x = 5$, which satisfies the condition $x > 4$.

Examples

Example 5

Solve $8^{\log_4(x+1)} = 27$.

Solution

First note restrictions on x; since $x+1 > 0$, then $x > -1$.

We know $c^{\log_c(n)} = n$, $n > 0$, but the bases must be the same.

The logarithm is base 4 and we can express 8 as a power base 4; that is, $8 = 4^{\frac{3}{2}}$.

$$\begin{aligned} 8^{\log_4(x+1)} &= 27 \\ \left(4^{\frac{3}{2}}\right)^{\log_4(x+1)} &= 27 \\ \left(4^{\log_4(x+1)}\right)^{\frac{3}{2}} &= 27 \quad \text{since } (x^a)^b = (x^b)^a = x^{ab} \end{aligned}$$

Examples

Example 5

Solve $8^{\log_4(x+1)} = 27$.

Solution

$$\left(4^{\log_4(x+1)}\right)^{\frac{3}{2}} = 27$$

Using the property $c^{\log_c(n)} = n, n > 0$, we know $4^{\log_4(x+1)} = x + 1$.
So

$$\begin{aligned}(x+1)^{\frac{3}{2}} &= 27\\ \left((x+1)^{\frac{3}{2}}\right)^{\frac{2}{3}} &= 27^{\frac{2}{3}}\\ x+1 &= \left(\sqrt[3]{27}\right)^2\\ x+1 &= 9\\ x &= 8 \text{ which satisfies } x > -1\end{aligned}$$

Therefore, $x = 8$.

Laws of Logarithms

Logarithm of a Product

Product Law
$\log_c(xy) = \log_c(x) + \log_c(y)$, where $c > 0, c \neq 1, x > 0, y > 0$.

This is the logarithmic form of the exponent law $(c^m)(c^n) = c^{m+n}$.

Proof:
Let $\log_c(x) = m$ and $\log_c(y) = n$.
Then, $c^m = x$ and $c^n = y$.
Now,

$$\begin{aligned}\log_c(xy) &= \log_c(c^m \cdot c^n) && \text{since } x = c^m \text{ and } y = c^n\\ &= \log_c(c^{m+n})\\ &= m + n && \text{but } m = \log_c(x) \text{ and } n = \log_c(y)\\ &= \log_c(x) + \log_c(y)\end{aligned}$$

$\therefore \log_c(xy) = \log_c(x) + \log_c(y)$, as required.

Laws of Logarithms

Logarithm of a Quotient

Quotient Law

$\log_c\left(\frac{x}{y}\right) = \log_c(x) - \log_c(y)$, where $c > 0, c \neq 1, x > 0, y > 0$.

This is the logarithmic form of the exponent law $\frac{c^m}{c^n} = c^{m-n}$.

Proof:
Let $\log_c(x) = m$ and $\log_c(y) = n$.
Rewriting each in exponential form gives $c^m = x$ and $c^n = y$.
Now,

$$\begin{aligned} \log_c\left(\frac{x}{y}\right) &= \log_c\left(\frac{c^m}{c^n}\right) && \text{since } x = c^m \text{ and } y = c^n \\ &= \log_c(c^{m-n}) \\ &= m - n && \text{but } m = \log_c(x) \text{ and } n = \log_c(y) \\ &= \log_c(x) - \log_c(y) \end{aligned}$$

$\therefore \log_c\left(\frac{x}{y}\right) = \log_c(x) - \log_c(y)$, as required.

Laws of Logarithms

Logarithm of a Power

Power Law

$\log_c(x^n) = n\log_c(x)$, where $c > 0, c \neq 1, x > 0$

This is the logarithmic form of the exponent law $(c^m)^n = c^{mn}$.

Proof:
Let $\log_c(x) = m$. Then, $c^m = x$.
Now,

$$\begin{aligned} \log_c(x^n) &= \log_c\left((c^m)^n\right) && \text{since } x = c^m \text{ and } y = c^n \\ &= \log_c(c^{mn}) \\ &= mn && \text{but } m = \log_c(x) \\ &= n\log_c(x) \end{aligned}$$

$\therefore \log_c(x^n) = n\log_c(x)$, as required.

Examples

Example 6

Simplify each expression to a single logarithm.

a. $\log_3(x) + \log_3(x-1) - \log_3(2x)$

b. $\log(2p) - 2\log(q) + 3\log(pq)$

Solution

a. $\log_3(x) + \log_3(x-1) - \log_3(2x)$

$$\begin{aligned} &= \log_3\big(x(x-1)\big) - \log_3(2x) \\ &= \log_3\left(\frac{x(x-1)}{2x}\right) \\ &= \log_3\left(\frac{x-1}{2}\right) \end{aligned}$$

Note that $x > 1$.

b. $\log(2p) - 2\log(q) + 3\log(pq)$

$$\begin{aligned} &= \log(2p) - \log(q^2) + \log\big((pq)^3\big) \\ &= \log\left(\frac{2p}{q^2}\right) + \log(p^3q^3) \\ &= \log\left(\frac{2p}{q^2}(p^3q^3)\right) \\ &= \log(2p^4q) \end{aligned}$$

Note that $p > 0$ and $q > 0$.

Examples

Example 7

Express $\log_2\left(\frac{4x^2}{\sqrt{y}}\right)$ in terms of $\log_2(x)$ and $\log_2(y)$.

Solution

$$\begin{aligned} \log_2\left(\frac{4x^2}{\sqrt{y}}\right) &= \log_2(4x^2) - \log_2(\sqrt{y}) \\ &= \log_2(4) + \log_2(x^2) - \log_2\left(y^{\frac{1}{2}}\right) \\ &= 2 + 2\log_2(x) - \frac{1}{2}\log_2(y) \end{aligned}$$

Note: To express $\log_2\left(\frac{4x^2}{\sqrt{y}}\right)$ in terms of $\log_2(x)$ and $\log_2(y)$, implies $x > 0$ and $y > 0$.

Restrictions on the variables for $\log_2\left(\frac{4x^2}{\sqrt{y}}\right)$ are $y > 0$ and $x \neq 0$.

Restrictions on the variables for $2 + 2\log_2(x) - \frac{1}{2}\log_2(y)$ are $x > 0$ and $y > 0$.

In order to apply the power law to express $\log_2(x^2)$ as $2\log_2(x)$, x must be greater than zero.

Examples

Example 8

Simplify and evaluate $2\log_3(12) - \log_3(6) - 3\log_3(2)$.

Solution

$$\begin{aligned} 2\log_3(12) - \log_3(6) - 3\log_3(2) &= \log_3\left(12^2\right) - \log_3(6) - \log_3\left(2^3\right) \\ &= \log_3\left(\frac{144}{6}\right) - \log_3(8) \\ &= \log_3\left(\frac{144}{6(8)}\right) \\ &= \log_3(3) \\ &= 1 \end{aligned}$$

Examples

Example 9

Use logarithms to solve $5^x = 30$.

Solution

Method 1:

$$\begin{aligned} 5^x &= 30 \\ \log(5^x) &= \log(30) \\ x\log(5) &= \log(30) \\ x &= \frac{\log(30)}{\log(5)} \\ x &\approx 2.113 \end{aligned}$$

Check: $5^{2.113} = 29.98635...$

Method 2:

$5^x = 30$ can be re-expressed in logarithmic form

$$\log_5(30) = x$$

Thus, $x = \log_5(30)$. We can also conclude that $\log_5(30) = \dfrac{\log(30)}{\log(5)}$, the solution obtained in our first approach.

Laws of Logarithms

Change of Base Formula

$\log_c(x) = \dfrac{\log_a(x)}{\log_a(c)}$, where a and $c > 0$, a and $c \neq 1, x > 0$

Proof:

Let $\log_c(x) = n$, so $c^n = x$.

Then,

$$\begin{aligned} \log_a(c^n) &= \log_a(x) \\ n\log_a(c) &= \log_a(x) \\ n &= \frac{\log_a(x)}{\log_a(c)} \quad \text{but } n = \log_c(x) \\ \log_c(x) &= \frac{\log_a(x)}{\log_a(c)} \end{aligned}$$

This formula allows us to convert any logarithm to a different base. For example, $\log_3(20)$ can be calculated using the common log function [log] on the calculator by converting the base 3 logarithm to base 10.

$$\log_3(20) = \frac{\log(20)}{\log(3)}$$

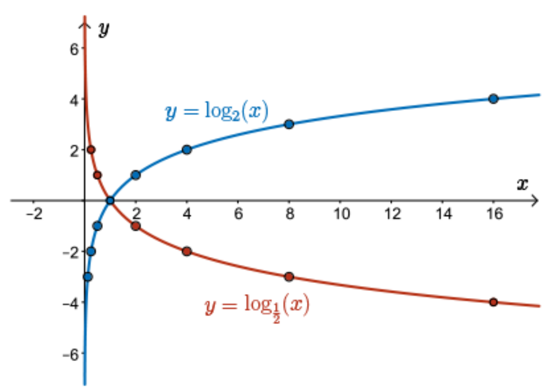
Therefore, $\log_3(20) \approx 2.7268$.

Laws of Logarithms

In the previous module, Logarithmic Functions, we determined graphically that $\log_{\frac{1}{2}}(x) = -\log_2(x)$, since the graph of $y = \log_{\frac{1}{2}}(x)$ is a reflection of $y = \log_2(x)$ in the x-axis.

We can prove this algebraically using the change of base formula.

$$\begin{aligned} \log_{\frac{1}{2}}(x) &= \frac{\log_2(x)}{\log_2\left(\frac{1}{2}\right)} \\ &= \frac{\log_2(x)}{-1} \\ &= -\log_2(x) \end{aligned}$$



Laws of Logarithms

Challenge Problem

If $\log_2(x), 1+\log_4(x)$, and $\log_8(4x)$ are consecutive terms of a geometric sequence, determine the possible values of x.

Note:

A geometric sequence is a sequence of the form a, ar, ar^2, ar^3, $\ldots$, ar^{n-1}, $\ldots$ where the ratio of any term to the preceding term is constant $\left(\frac{t_{n+1}}{t_n} = r,\ r \neq 0,\ n \geq 1,\ n \in \mathbb{Z}\right)$, $a \neq 0$.

Solution

Since $\log_2(x), 1+\log_4(x)$, and $\log_8(4x)$ are consecutive terms of a geometric sequence, then

$$\frac{1+\log_4(x)}{\log_2(x)} = \frac{\log_8(4x)}{1+\log_4(x)}$$

Now, $x > 0$ and $4x > 0$, and thus $x > 0$.

Also, $\log_2(x) \neq 0$ and $1+\log_4(x) \neq 0$. Therefore, $x \neq 1$ or $\frac{1}{4}$.

To solve this equation, we will convert $\log_4(x)$ and $\log_8(4x)$ to base 2 logarithms to work with a common base.

Laws of Logarithms

Challenge Problem

If $\log_2(x), 1+\log_4(x)$, and $\log_8(4x)$ are consecutive terms of a geometric sequence, determine the possible values of x.

Solution

$$\frac{1+\log_4(x)}{\log_2(x)} = \frac{\log_8(4x)}{1+\log_4(x)}$$

$$\log_4(x) = \frac{\log_2(x)}{\log_2(4)} = \frac{\log_2(x)}{2} = \frac{1}{2}\log_2(x)$$

$$\log_8(4x) = \frac{\log_2(4x)}{\log_2(8)} = \frac{\log_2(4x)}{3} = \frac{\log_2(4)+\log_2(x)}{3} = \frac{2}{3} + \frac{1}{3}\log_2(x)$$

So we must solve

$$\frac{1+\frac{1}{2}\log_2(x)}{\log_2(x)} = \frac{\frac{2}{3}+\frac{1}{3}\log_2(x)}{1+\frac{1}{2}\log_2(x)}$$

Laws of Logarithms

Challenge Problem

If $\log_2(x), 1+\log_4(x)$, and $\log_8(4x)$ are consecutive terms of a geometric sequence, determine the possible values of x.

Solution

$$\frac{1+\frac{1}{2}\log_2(x)}{\log_2(x)} = \frac{\frac{2}{3}+\frac{1}{3}\log_2(x)}{1+\frac{1}{2}\log_2(x)}$$

Let $a = \log_2(x)$ and solve for a.

$$\begin{aligned}
\frac{1+\frac{1}{2}a}{a} &= \frac{\frac{2}{3}+\frac{1}{3}a}{1+\frac{1}{2}a} \\
\frac{2\left(1+\frac{1}{2}a\right)}{2a} &= \frac{6\left(\frac{2}{3}+\frac{1}{3}a\right)}{6\left(1+\frac{1}{2}a\right)} \\
\frac{2+a}{2a} &= \frac{4+2a}{6+3a} \\
(2+a)(6+3a) &= 2a(4+2a) \\
12+12a+3a^2 &= 8a+4a^2 \\
a^2-4a-12 &= 0 \\
(a-6)(a+2) &= 0
\end{aligned}$$

So $a = 6, -2$.

Since $a = \log_2(x)$,

$$\begin{aligned}
\log_2(x) &= 6 \quad &\text{or}\quad \log_2(x) &= -2 \\
x &= 2^6 & x &= 2^{-2} \\
x &= 64 & x &= \frac{1}{4}
\end{aligned}$$

Laws of Logarithms

Challenge Problem

If $\log_2(x), 1+\log_4(x)$, and $\log_8(4x)$ are consecutive terms of a geometric sequence, determine the possible values of x.

Solution

But only $x = 64$ satisfies the restriction $x > 0, x \neq 1, \frac{1}{4}$.

Therefore, the three terms, $\log_2(x), 1+\log_4(x)$, and $\log_8(4x)$, form a geometric sequence when $x = 64$.

$$\log_2(64) = 6, \quad 1+\log_4(64) = 1+3 = 4, \quad \text{and} \quad \log_8(4\cdot 64) = \log_8(256) = \frac{8}{3}$$

The three terms of the sequence are 6, 4, $\frac{8}{3}$, which have a common ratio of $\frac{2}{3}$.

Note:

When $x = \frac{1}{4}$, the three terms of the sequence (i.e. -2, 0, and 0) have a common ratio of 0. A geometric sequence cannot have a common ratio of 0.

A geometric sequence cannot have a common ratio of 0.

Summary

- The logarithmic function, $y = \log_c(x)$, is the inverse of the exponential function, $y = c^x$, where $c > 0,\ c \neq 1$.
- The statement $\log_c(m) = n$ is equivalent to the statement $m = c^n$.
- The logarithm, $\log_c(x)$, is defined only when $c > 0$, $c \neq 1$ and $x > 0$.
- A common logarithm is a logarithm base 10. When the base is not provided, such as $\log(x)$, it is assumed to be a common logarithm.
- The following statements hold true for logarithms with $c > 0$ and $c \neq 1$:
  - $c^{\log_c(n)} = n$, $n > 0$
  - $\log_c(c^n) = n$
  - $\log_c(1) = 0$
  - $\log_c(m) = \log_c(n)$ if and only if $m = n$, m and $n > 0$
- For $c > 0$, $c \neq 1$, $x > 0$, $y > 0$,
  - $\log_c(xy) = \log_c(x) + \log_c(y)$ (Product Law)
  - $\log_c\left(\frac{x}{y}\right) = \log_c(x) - \log_c(y)$ (Quotient Law)
  - $\log_c(x^n) = n\log_c(x)$ (Power Law)
- The formula for converting a logarithm from one base to another is $\log_c(x) = \frac{\log_a(x)}{\log_a(c)}$. This formula is often used to convert to base 10.

All of the above properties and laws of logarithms can be used to simplify logarithmic expression and solve exponential and logarithmic equations.