The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Solving Exponential Equations

Introduction

In order to solve problems involving exponential functions, we will need the skills necessary to solve exponential equations.

Exponential equations are equations in which the variable occurs in the exponent.

In This Module

- We will discuss methods of solving exponential equations using the laws of exponents to obtain common bases.

Examples

Example 1

Solve $2^{x-3} = 8\sqrt{2}$.

Solution

$$\begin{aligned}2^{x-3} &= 8\sqrt{2}\\ 2^{x-3} &= 2^3 \cdot 2^{\frac{1}{2}}\\ 2^{x-3} &= 2^{3+\frac{1}{2}}\\ 2^{x-3} &= 2^{\frac{7}{2}}\end{aligned}$$

If two powers with the same base are equivalent, then their exponents must be equivalent.

Equating the exponents,

$$\begin{aligned}x-3 &= \frac{7}{2}\\ x &= \frac{13}{2}\end{aligned}$$

Examples

Example 2

Solve $3^{2x+1} = \frac{1}{81}$.

Solution

$$
\begin{aligned}
3^{2x+1} &= \frac{1}{81} \\
3^{2x+1} &= \frac{1}{3^4} \\
3^{2x+1} &= 3^{-4}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
2x + 1 &= -4 \\
x &= -\frac{5}{2}
\end{aligned}
$$

Answers can be verified by substituting the value of the variable into the left and/or right sides of the equation to determine equality.

$$
\begin{aligned}
\text{L.S.} &= 3^{2x+1} \\
&= 3^{2(-5/2)+1} \\
&= 3^{-5+1} \\
&= 3^{-4} \\
&= \frac{1}{81}
\end{aligned}
\qquad
\text{R.S.} = \frac{1}{81}
$$

Since $\text{L.S.} = \text{R.S.}$, we have that $x = -\frac{5}{2}$.

Examples

Example 3

Solve $25^{x-2} = 125^{2x-4}$.

Solution

$$
\begin{aligned}
25^{x-2} &= 125^{2x-4} \\
(5^2)^{x-2} &= (5^3)^{2x-4} \\
5^{2x-4} &= 5^{6x-12}
\end{aligned}
$$

Equating the exponents,

$$
\begin{aligned}
2x - 4 &= 6x - 12 \\
-4x &= -8 \\
x &= 2
\end{aligned}
$$

Verifying,

$$
\begin{aligned}
\text{L.S.} &= 25^{x-2} \\
&= 25^{2-2} \\
&= 25^0 \\
&= 1
\end{aligned}
\qquad
\begin{aligned}
\text{R.S.} &= 125^{2x-4} \\
&= 125^{2(2)-4} \\
&= 125^0 \\
&= 1
\end{aligned}
$$

Since $\text{L.S.} = \text{R.S.}$, we have that $x = 2$.

Examples

Example 4

Solve $4^x(8^{x-3}) = \dfrac{1}{16^{2-x}}$.

Solution

$$\begin{aligned} 4^x(8^{x-3}) &= \frac{1}{16^{2-x}} \\ (2^2)^x(2^3)^{x-3} &= (2^{-4})^{2-x} \\ (2^{2x})(2^{3x-9}) &= 2^{-8+4x} \\ 2^{5x-9} &= 2^{-8+4x} \end{aligned}$$

Equating the exponents,

$$\begin{aligned} 5x - 9 &= -8 + 4x \\ x &= 1 \end{aligned}$$

Examples

Example 5

Solve $8^{x^2} = 4^{-5x+4}$.

Solution

$$\begin{aligned} 8^{x^2} &= 4^{-5x+4} \\ (2^3)^{x^2} &= (2^2)^{-5x+4} \\ 2^{3x^2} &= 2^{-10x+8} \end{aligned}$$

Equating the exponents,

$$3x^2 = -10x + 8$$

We now solve the quadratic,

$$\begin{aligned} 3x^2 + 10x - 8 &= 0 \\ (3x-2)(x+4) &= 0 \end{aligned}$$

so, $x = \dfrac{2}{3}$ and $x = -4$.

Therefore, the equation $8^{x^2} = 4^{-5x+4}$ has two solutions: $x = \dfrac{2}{3}$ and $x = -4$.

Examples

Example 6

Solve $3^{2x} - 10(3^x) + 9 = 0$.

Solution

This equation is quadratic in form.

The first term is $3^{2x} = (3^x)^2$. Let $a = 3^x$.

$$\begin{aligned}(3^x)^2 - 10(3^x) + 9 &= 0\\ a^2 - 10a + 9 &= 0\\ (a-1)(a-9) &= 0\\ a = 1 \text{ or } a &= 9\end{aligned}$$

Then, let $a = 3^x$. Thus, $3^x = 1$ or $3^x = 9$.

This means that $x = 0$ or $x = 2$.

Alternatively, working with the original equation,

$$\begin{aligned}(3^x)^2 - 10(3^x) + 9 &= 0\\ (3^x - 1)(3^x - 9) &= 0\\ 3^x = 1 \ \text{ or } 3^x &= 9\\ x = 0 \ \text{ or } x &= 2\end{aligned}$$

Examples

Example 7

Solve $5^x = 40$.

Trial and Error Solution

We know that

- $5^2 = 25$ and $5^3 = 125$ so $2 < x < 3$
- since 40 is much closer to 25 than 125, x will be closer in value to 2

Checking $x = 2.2$, gives

$$5^{2.2} \approx 34.4932$$

Checking $x = 2.3$, gives

$$5^{2.3} \approx 40.5164$$

Checking $x = 2.28$, gives

$$5^{2.28} \approx 39.2330$$

Checking $x = 2.29$, gives

$$5^{2.29} \approx 39.8695$$

Thus, we may conclude that $x \approx 2.29$, to within two decimal places accuracy.

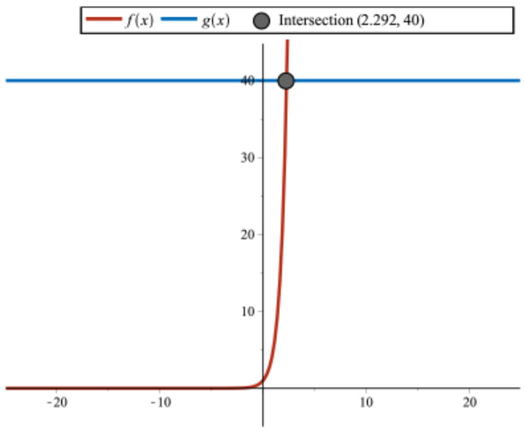
Examples

Example 7

Solve $5^x = 40$.

Graphing Technology Solution

Method 2: This equation can be solved using graphing technology.
Let $f(x) = 5^x$ and $g(x) = 40$. Graph each function.
By identifying the point of intersection of the two functions, we can solve the equation $5^x = 40$.
The x-coordinate of the point of intersection provides the solution to the equation since $f(x) = g(x)$ for this specific value of x.
The two functions intersect at approximately $(2.29, 40)$.
Therefore, x is approximately 2.29.



Examples

Example 8

Determine the x-intercept of the function $g(x) = \frac{1}{2}f(x) - 6$, where $f(x)$ is an exponential function of the form $f(x) = a(c^x),\ a \neq 0,\ c > 0,\ c \neq 1$, such that $f(x+2) - f(x+1) = 12f(x)$ for all $x \in \mathbb{R}$ and $f(-1) = \frac{3}{2}$.

Solution

We first determine the equation of the function, $f(x) = a(c^x)$ by finding the two unknowns a and c.

We are given

$$f(x+2) - f(x+1) = 12f(x)$$

So

$$a(c^{x+2}) - a(c^{x+1}) = 12a(c^x)$$

Since $a \neq 0$, we can divide both sides of the equation by a to obtain

$$c^{x+2} - c^{x+1} = 12c^x$$

Examples

Example 8

Determine the x-intercept of the function $g(x) = \frac{1}{2}f(x) - 6$, where $f(x)$ is an exponential function of the form $f(x) = a(c^x),\ a \neq 0,\ c > 0,\ c \neq 1$, such that $f(x+2) - f(x+1) = 12f(x)$ for all $x \in \mathbb{R}$ and $f(-1) = \frac{3}{2}$.

Solution

$$c^{x+2} - c^{x+1} = 12c^x$$

By applying the exponent law, $c^{m+n} = (c^m)(c^n)$, to each term on the left side of the equation, we see that c^x is a factor of each term, giving

$$\begin{aligned}(c^x)(c^2) - (c^x)(c) &= 12c^x \\ c^x(c^2 - c) &= 12c^x\end{aligned}$$

Now, $c^x \neq 0$ since $c > 0$, so

$$\begin{aligned}c^2 - c &= 12 \\ c^2 - c - 12 &= 0 \\ (c-4)(c+3) &= 0 \\ c = 4 \text{ or } c &= -3\end{aligned}$$

However, $f(x)$ is an exponential function so $c > 0$, therefore $c = 4$.

Examples

Example 8

Determine the x-intercept of the function $g(x) = \frac{1}{2}f(x) - 6$, where $f(x)$ is an exponential function of the form $f(x) = a(c^x),\ a \neq 0,\ c > 0,\ c \neq 1$, such that $f(x+2) - f(x+1) = 12f(x)$ for all $x \in \mathbb{R}$ and $f(-1) = \frac{3}{2}$.

Solution

We now know that $f(x) = a(4^x)$. Using $f(-1) = \frac{3}{2}$, we may solve for a.

$$\begin{aligned}a(4^{-1}) &= \frac{3}{2} \\ \frac{a}{4} &= \frac{3}{2} \\ \therefore a &= 6\end{aligned}$$

Thus, $f(x) = 6(4^x)$ and

$$\begin{aligned}g(x) &= \frac{1}{2}f(x) - 6 \\ &= \frac{1}{2}\left(6(4^x)\right) - 6 \\ &= 3(4^x) - 6\end{aligned}$$

Therefore, $g(x) = 3(4^x) - 6$.

Examples

Example 8

Determine the x-intercept of the function $g(x) = \frac{1}{2}f(x) - 6$, where $f(x)$ is an exponential function of the form $f(x) = a(c^x),\ a \neq 0,\ c > 0,\ c \neq 1$, such that $f(x+2) - f(x+1) = 12f(x)$ for all $x \in \mathbb{R}$ and $f(-1) = \frac{3}{2}$.

Solution

$$g(x) = 3(4^x) - 6$$

To determine the x-intercept, we set $g(x) = 0$.

$$\begin{aligned} 0 &= 3(4^x) - 6 \\ 3(4^x) &= 6 \\ 4^x &= 2 \\ 2^{2x} &= 2 \\ \therefore 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

Therefore, the x-intercept of $g(x) = 3(4^x) - 6$ is at $\left(\frac{1}{2}, 0\right)$.

Summary

- Some exponential equations can be solved algebraically by expressing the power on each side of the equation with a common base and using the property that $x = y$ when $c^x = c^y$. Exponent laws may need to be applied to simplify the exponential expressions first.
- Exponential equations with powers that cannot be expressed using a common base can be solved using graphing technology, or an approximate solution can be obtained using a systematic trial and error method.