Interval Notation And Definitions

In This Module

Sometimes when solving mathematical problems, the solution is a set of numbers lying in an interval.
There are several ways to describe these intervals.

- We can write the solution algebraically using set notation.

$$\{x \mid x < 8, x \in \mathbb{R}\}$$

- We can display the solution graphically on a number line.

- We can use a special notation called interval notation.

$$x \in (-\infty, -1] \cup [1, +\infty),\ x \in \mathbb{R}$$

Describing Solutions Algebraically

A solution can be described algebraically using various symbols.

Symbol	Meaning	Word Example	Algebraic Solution
$<$	less than	x less than 8	$\{x \mid x < 8,\ x \in \mathbb{R}\}$
$>$	greater than	x greater than -2	$\{x \mid x > -2,\ x \in \mathbb{R}\}$
$\geq$	greater than or equal to	x greater than or equal to -20	$\{x \mid x \geq -20,\ x \in \mathbb{R}\}$
$\leq$	less than or equal to	x less than or equal to 100	$\{x \mid x \leq 100,\ x \in \mathbb{R}\}$
$\neq$	not equal to	x not equal to 11.7	$\{x \mid x \neq 11.7,\ x \in \mathbb{R}\}$

Describing Solutions Algebraically

Example 1

Describe the set of real numbers from -5 to 3 that includes -5, but does not include 3.

Solution

$$\{x \mid -5 \le x < 3,\ x \in \mathbb{R}\}$$

We read this, "the set of all x such that -5 is less than or equal to x, which is less than 3, and x is an element of the set of real numbers."

Example 2

Describe the set of real numbers which does not include 1.

Solution

$$\{x \mid x \neq 1,\ x \in \mathbb{R}\}$$

We read this, "the set of all x such that x is not equal to 1 and x is an element of the set of real numbers."

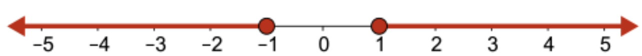
Describing Solutions Graphically

An interval can be described graphically using a number line.

A solid dot means that the endpoint is included in the interval.
An open dot means that the endpoint is excluded from the interval.
The example shown illustrates the solution which could be written in set notation as

$$\{x \mid -2 < x \le 3,\ x \in \mathbb{R}\}$$

Another interval is illustrated on the following number line:



Using set notation, this solution would be written

$$\{x \mid x \le -1 \text{ or } x \ge 1,\ x \in \mathbb{R}\}$$

Note
The use of the word or tells us that there is a union of two solutions. In this case, x can take on values less than or equal to -1, or x can take on values greater than or equal to 1.
Alternatively, x cannot take on values between -1 and 1.

Describing Solutions Using Interval Notation

We have described the solution, "x greater than -2 and x less than or equal to 3" using set notation

$$\{x \mid -2 < x \leq 3,\ x \in \mathbb{R}\}$$

and graphically with a number line

The same solution can be described using interval notation as follows:

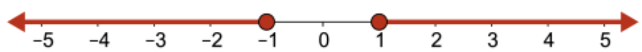
$$x \in (-2, 3],\ x \in \mathbb{R}$$

This interval goes from the smallest number, -2, to the largest number, 3.
The round bracket to the left of -2 indicates that -2 is excluded from the interval.
The square bracket to the right of 3 indicates that 3 is included in the interval.

If an interval is written $x \in [a, b),\ x \in \mathbb{R}$, then a would be included in the interval and b would be excluded from it.
If the interval were written $x \in (a, b),\ x \in \mathbb{R}$, then neither endpoint would be included in the interval.
If the interval were written $x \in [a, b],\ x \in \mathbb{R}$, then both endpoints would be included in the interval.

Describing Solutions Using Interval Notation

We also illustrated the solution, "x less than or equal to -1 or x greater than or equal to 1" on a number line.



Algebraically, using set notation, this solution would be written $\{x \mid x \leq -1 \text{ or } x \geq 1,\ x \in \mathbb{R}\}$.
Using interval notation, this solution would be written

$$x \in (-\infty, -1] \cup [1, +\infty),\ x \in \mathbb{R}$$

- The solution includes all numbers less than or equal to -1; however, we can never reach $-\infty$. Therefore, $-\infty$ is not included in the interval.
- The solution includes all numbers greater than or equal to 1; however, we can never reach $+\infty$. Therefore, $+\infty$ is not included in the interval.
- The symbol $\cup$ represents the union of the two sets.

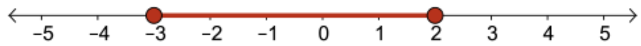
Examples Involving Intervals

Example 3

Represent "all real numbers from -3 to 2, inclusive" in the following ways: algebraically, on a number line, and using interval notation.

Solution

Algebraically, in set notation, we write $\{x \mid -3 \leq x \leq 2,\ x \in \mathbb{R}\}$.
On a number line, the solution is shown as



Using interval notation, this solution would be written $x \in [-3, 2],\ x \in \mathbb{R}$.

Examples Involving Intervals

Example 4

Represent "all real numbers except -1 and 4." in the following ways: algebraically, on a number line, and using interval notation.

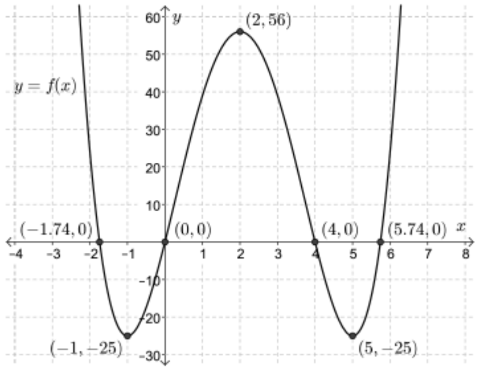
Solution

Algebraically, in set notation, we write $\{x \mid x \neq -1,\ x \neq 4,\ x \in \mathbb{R}\}$.
On a number line, the solution is shown as

Using interval notation, this solution would be written $x \in (-\infty, -1) \cup (-1, 4) \cup (4, +\infty),\ x \in \mathbb{R}$.

Note: $+\infty$ will sometimes be written without the $+$ symbol, simply as ∞.

An Illustrative Example

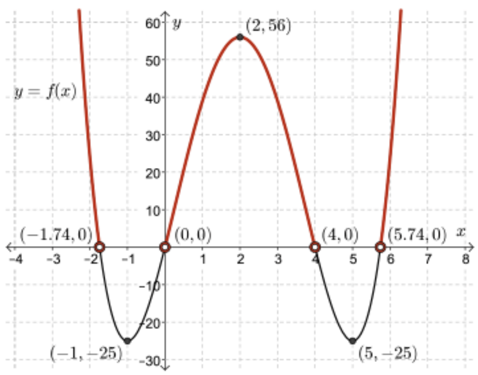


The graph shows some function $y = f(x)$.
Note the following about the graph.

- There are four x-intercepts: $x = -1.74$, $x = 0$, $x = 4$, and $x = 5.74$.
- Between $x = -1.74$ and $x = 0$, the graph reaches a local minimum point at $(-1, -25)$.
- Between $x = 0$ and $x = 4$, the graph reaches a local maximum point at $(2, 56)$.
- Between $x = 4$ and $x = 5.74$, the graph reaches a local minimum point at $(5, -25)$.
- As values of x get smaller and smaller, the value of the function gets larger and larger.
- As values of x get larger and larger, the value of the function gets larger and larger.

Note
At this time, do not be concerned with where the points on the graph came from. We will use the graph to illustrate some definitions which will be used throughout the course. We will also use the graph to illustrate a use for interval notation.

Positive Intervals



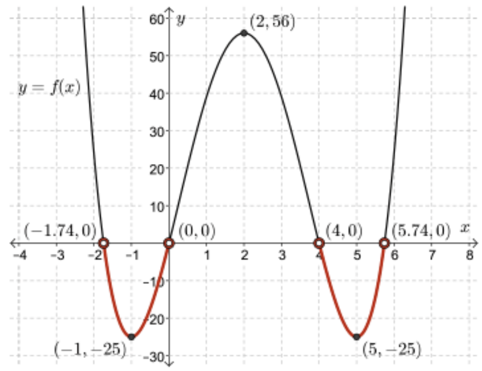
A function $y = f(x)$ is **positive on an interval** if the value of $f(x)$ is greater than 0 (i.e., $y > 0$) for all values of x in the interval.
Geometrically, the graph of the function resides **above** the x-axis in the given interval.

In our example, $f(x)$ is above the x-axis for values of $x < -1.74$, for $0 < x < 4$, and for $x > 5.74$.
In interval notation, we can write the positive interval

$$x \in (-\infty, -1.74) \cup (0, 4) \cup (5.74, \infty)$$

Negative Intervals

A function $y = f(x)$ is **negative on an interval** if the value of $f(x)$ is less than 0 (i.e., $y < 0$) for all values of x in the interval.
Geometrically, the graph of the function resides **below** the x-axis in the given interval.

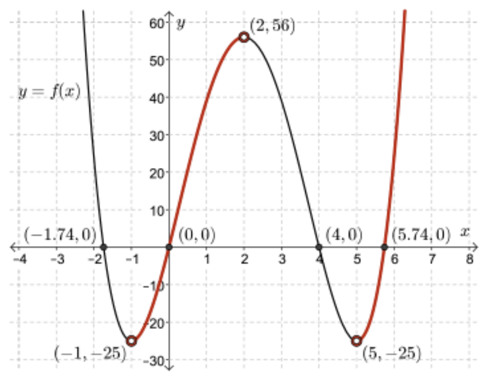


In our example, $f(x)$ is below the x-axis for $-1.74 < x < 0$ and $4 < x < 5.74$.
In interval notation, we can write the negative interval

$$x \in (-1.74, 0) \cup (4, 5.74)$$

Increasing Intervals

A function $f(x)$ is **increasing on an interval** if the value of $y = f(x)$ increases as the value of x increases.
More precisely, a function f is increasing on an interval, (a, b), if for every $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, $f(x_1) \leq f(x_2)$.
Geometrically, as one moves from left to right along the curve, the value of the y-coordinate never decreases.
The function is said to be **strictly increasing** if $f(x_1) < f(x_2)$ for $x_1 < x_2$ in the interval.

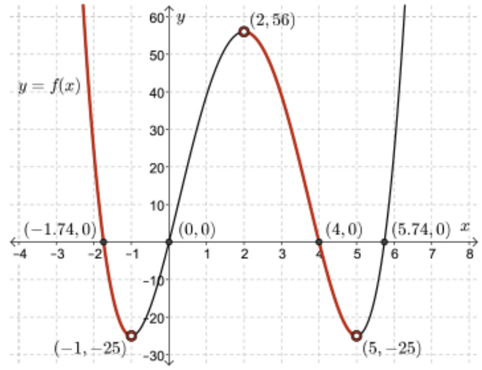


In our example, $f(x)$ is increasing for $-1 < x < 2$ and $x > 5$.
In interval notation, we can write the increasing interval

$$x \in (-1, 2) \cup (5, \infty)$$

Decreasing Intervals

A function $f(x)$ is **decreasing on an interval** if the value of $y = f(x)$ decreases as the value of x increases.
More precisely, a function f is decreasing on an interval, (a, b), if for every $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, $f(x_1) \geq f(x_2)$.
Geometrically, as one moves from left to right along the curve, the value of the y-coordinate never increases.
The function is said to be **strictly decreasing** if $f(x_1) > f(x_2)$ for $x_1 < x_2$ in the interval.



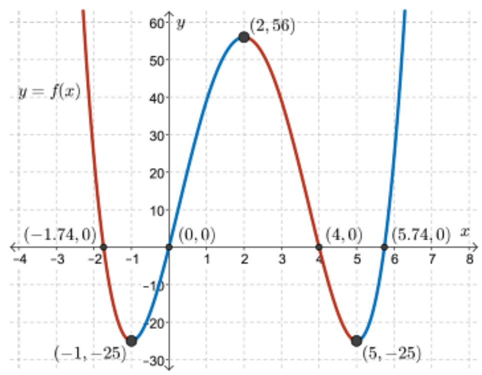
In our example, $f(x)$ is decreasing for $x < -1$ and $2 < x < 5$.
In interval notation, we can write the decreasing interval

$$x \in (-\infty, -1) \cup (2, 5)$$

Maxima and Minima

A **local maximum** is a point on a graph whose y value is *greater than or equal to* the y values of all other points near it. The function changes from increasing to decreasing, as x increases, at a local maximum.

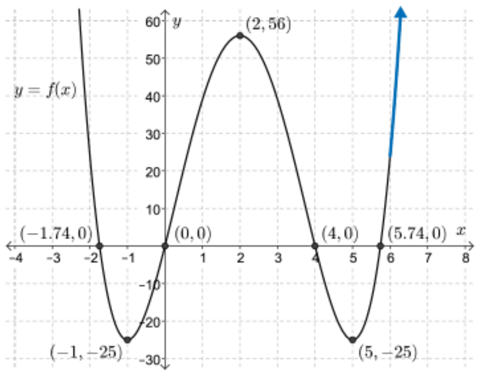
A **local minimum** is a point on a graph whose y value is *less than or equal to* the y values of all other points near it. The function changes from decreasing to increasing, as x increases, at a local minimum.



If there is more than one local minimum, we refer to them as local minima.
If there is more than one local maximum, we refer to them as local maxima.

On the graph, the increasing intervals are shown in blue and the decreasing intervals are shown in red.
At $(-1, -25)$, the function changes from decreasing to increasing, so $(-1, -25)$ is a local minimum.
At $(2, 56)$, the function changes from increasing to decreasing, so $(2, 56)$ is a local maximum.
At $(5, -25)$, the function changes from decreasing to increasing, so $(5, -25)$ is a local minimum.

End Behaviour



The end behaviour of a function refers to what happens to the function for extreme positive and negative values of x.

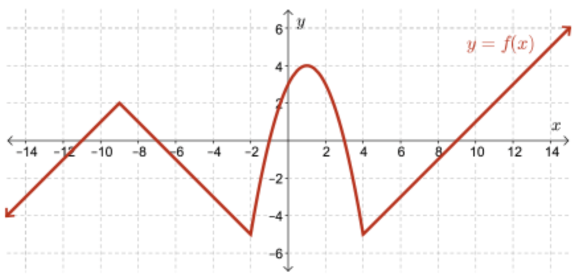
Mathematicians write "as $x \to \infty$" whenever x approaches very large values.
They write "as $x \to -\infty$" whenever x approaches very small values (which are large negative values).
We use "$\to$" for the word "approaches".

In our example, as $x \to -\infty$, the value of $y = f(x)$ tends to infinity.
Expressed mathematically, we write "as $x \to -\infty$, $f(x) \to \infty$" and read "as x approaches $-\infty$, $f(x)$ approaches ∞."

Similarly, as $x \to \infty$, the value of $y = f(x)$ tends to infinity.
Expressed mathematically, we write "as $x \to \infty$, $f(x) \to \infty$."

A Practice Example

Example 5



For the function $y = f(x)$, identify the x-intercepts, y-intercepts, positive and negative intervals, increasing and decreasing intervals, local maxima and local minima, and end behaviour. You may assume that the key points on the graph have integral coordinates.

Solution

x-intercepts:
$x = -11,\ x = -7,\ x = -1,\ x = 3,\ x = 9$

y-intercept:
$y = 3$

Positive Intervals:
$x \in (-11, -7) \cup (-1, 3) \cup (9, \infty)$

Negative Intervals:
$x \in (-\infty, -11) \cup (-7, -1) \cup (3, 9)$

Increasing Intervals:
$x \in (-\infty, -9) \cup (-2, 1) \cup (4, \infty)$

Decreasing Intervals:
$x \in (-9, -2) \cup (1, 4)$

Local Maxima:
$(-9, 2),\ (1, 4)$

Local Minima:
$(-2, -5),\ (4, -5)$

End Behaviour:
as $x \to -\infty$, $y \to -\infty$; as $x \to \infty$, $y \to \infty$

Summary

- We introduced the notion of solutions that lie in an interval.
- We have shown three ways to write such intervals: set notation, a number line, and interval notation.
- We made several definitions which will be used throughout this course.