The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

Relations And Functions

In This Module

- We will introduce relations and functions.
- We will represent relations and functions using set notation, mapping diagrams, and graphs.
- We will introduce the vertical line test.
- We will introduce domain and range.
- We will introduce function notation.
- We will introduce composite functions.

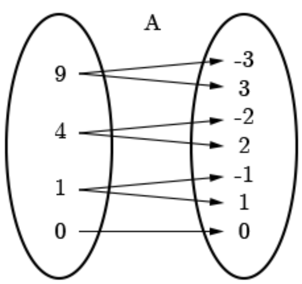
Relations and Functions

A **relation** is a set of ordered pairs.

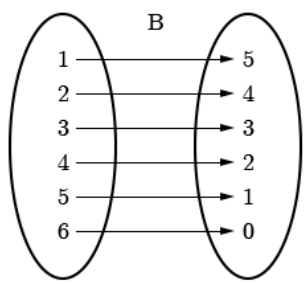
Let $A = \{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$.
Let $B = \{(1,5),(2,4),(3,3),(4,2),(5,1),(6,0)\}$.
The elements in each set have been listed using **set notation**.

A **function** is a relation, a set of ordered pairs (x, y), in which for every x value, there is only one y value.

By definition, A is not a function, whereas B is a function.
We can represent A and B using mapping diagrams.



Since at least one x-value maps to more than one y-value, A is not a function.

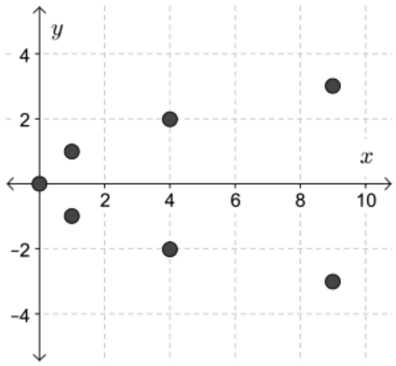


Since each x-value maps to exactly one y-value, B is a function.

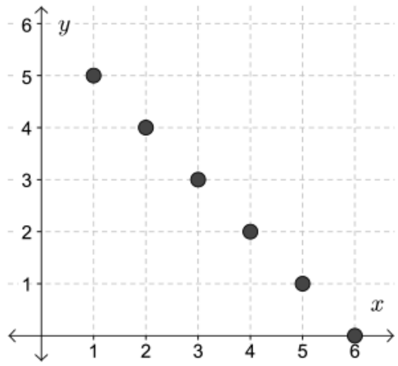
Representing Relations and Functions Graphically

We have represented A and B using set notation and mapping diagrams.
We can also represent A and B on a graph.

$A = \{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$



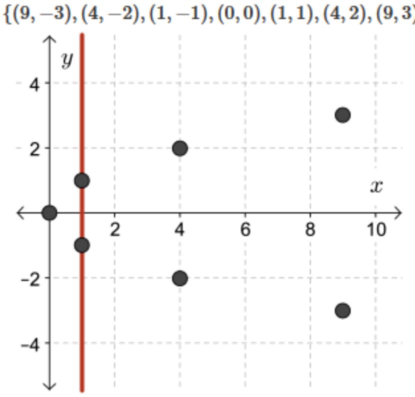
$B = \{(1,5),(2,4),(3,3),(4,2),(5,1),(6,0)\}$



The Vertical Line Test

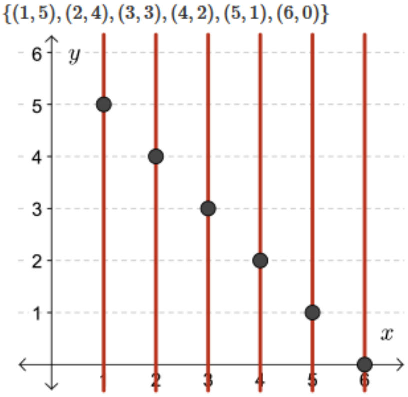
If a vertical line can be drawn anywhere on a graph so that it passes through two or more points on a relation, then the relation **is not** a function.
However, if no vertical line can be drawn that passes through more than one point on a relation, then the relation **is** a function.
This is called the **vertical line test**.

$A = \{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$



A vertical line can be drawn through the two points, $(1,-1)$ and $(1,1)$. Therefore, A is not a function.

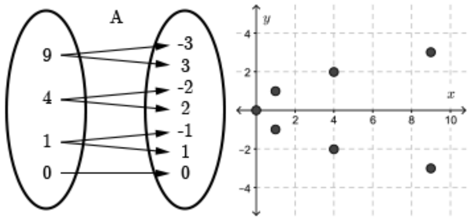
$B = \{(1,5),(2,4),(3,3),(4,2),(5,1),(6,0)\}$



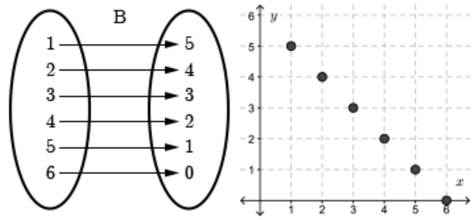
No vertical line can be drawn through any two points on B. Therefore, B is a function.

Domain and Range

$A = \{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$



$B = \{(1,5),(2,4),(3,3),(4,2),(5,1),(6,0)\}$



The set of all possible values of the independent variable, x, is called the **domain**.

The set of all possible values of the dependent variable, y, is called the **range**.

For
$A = \{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$
the domain is $\{0, 1, 4, 9\}$, and
the range is $\{-3, -2, -1, 0, 1, 2, 3\}$.

For
$B = \{(1,5),(2,4),(3,3),(4,2),(5,1),(6,0)\}$
the domain is $\{1, 2, 3, 4, 5, 6\}$, and
the range is $\{0, 1, 2, 3, 4, 5\}$.

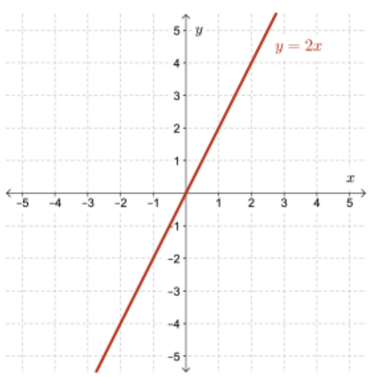
Using the terminology of domain and range, a function is a relation in which each element in the domain corresponds to exactly one element in the range.

Domain and Range

Example 1

State the domain and range of the following. State, with justification, whether or not each relation is a function.

a.



In set notation, the symbol, "$|$", is a mathematical short form for "such that" and the symbol, "$\in$", stands for "element of."

Solution

This relation is a function since it passes the vertical line test.
The domain is the set of real numbers. We write this as follows:

Domain is $\{x \mid x \in \mathbb{R}\}$

We read this, "The domain is the set of x values such that x is an element of the set of real numbers."

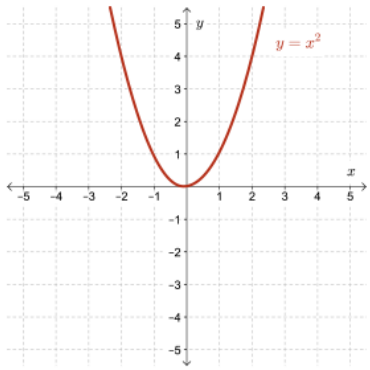
The range is the set of real numbers. We write this as follows:

Range is $\{y \mid y \in \mathbb{R}\}$

Domain and Range

Example 1

b.



Solution

This relation is a function since it passes the vertical line test.

The domain is the set of real numbers. There is no real number which cannot be squared. We write this as follows:

$$\text{Domain is } \{x \mid x \in \mathbb{R}\}$$

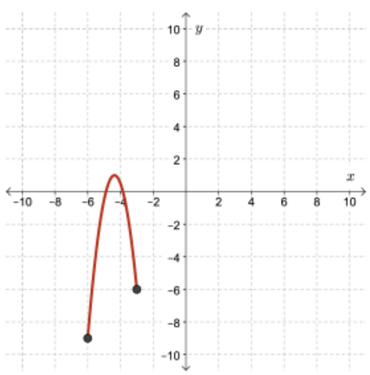
The range is the set of real numbers which are greater than or equal to zero since squaring any real number results in a real number that is positive or zero. We write this as follows:

$$\text{Range is } \{y \mid y \geq 0, y \in \mathbb{R}\}$$

Domain and Range

Example 1

c.



Solution

This relation is a function since it passes the vertical line test.

This example is different from the previous example in that there are definite starting and ending points on the graph.

It would appear from the graph that the domain is the set of real numbers from -6 to -3, inclusive. We write this as follows:

$$\text{Domain is } \{x \mid -6 \leq x \leq -3, x \in \mathbb{R}\}$$

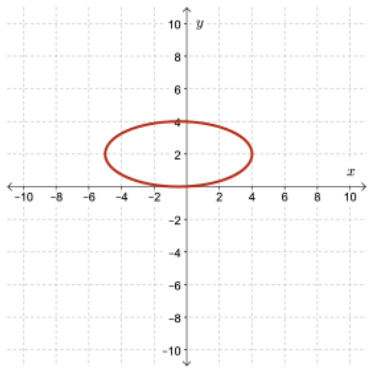
It would appear from the graph that the range is also restricted to real numbers between -9 and 1, inclusive. We write this as follows:

$$\text{Range is } \{y \mid -9 \leq y \leq 1, y \in \mathbb{R}\}$$

Domain and Range

Example 1

d.



Solution

This relation is not a function since it fails the vertical line test. For example, a vertical line drawn along the y-axis intersects the relation twice.

It would appear from the graph that the domain is the set of real numbers from -5 to 4, inclusive. We write this as follows:

Domain is $\{x \mid -5 \leq x \leq 4, x \in \mathbb{R}\}$

It would appear from the graph that the range is also restricted to real numbers between 0 and 4, inclusive. We write this as follows:

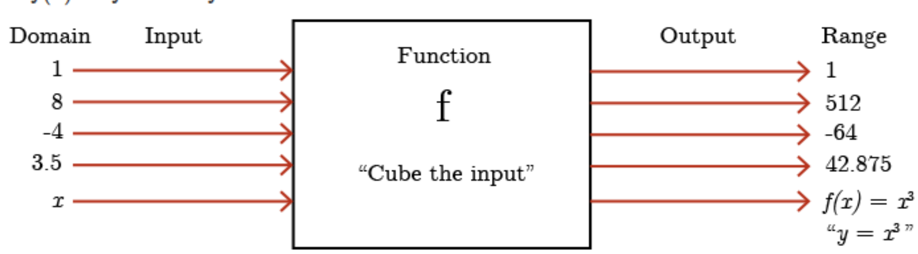
Range is $\{y \mid 0 \leq y \leq 4, y \in \mathbb{R}\}$

Function Notation

Think of a function as a machine. The machine inputs acceptable values from the domain, and then performs some operation to produce an output value in the range.

In the function machine illustrated below, an input is provided and the function cubes the input to produce an output. We use a special notation called function notation to represent this. In this case, $f(x) = x^3$.

We read $f(x)$ as "f at x" or "f of x".



So, $f(8)$ tells the function to cube 8 producing an output of 512. Therefore, $f(8) = 512$.

$f(x) = x^3$ can be written as the equation $y = x^3$; $f(x)$ and y are interchangeable.

Function notation is helpful in illustrating the value that is being substituted in the function for x.

If $f(x) = x^3$, then we know that $f(3.5)$ is asking us to evaluate the function when $x = 3.5$. It follows that $f(3.5) = 3.5^3 = 42.875$.

Function Notation

Example 2

Let $f(x) = 2x + 3$ and $g(x) = x^2 + 4x$.

a. Evaluate

i. $f(-6)$

ii. $g(-3)$

Solution

i. Since $f(x) = 2x + 3$,

$$\begin{aligned} f(-6) &= 2(-6) + 3 \\ &= -9 \end{aligned}$$

ii. Since $g(x) = x^2 + 4x$,

$$\begin{aligned} g(-3) &= (-3)^2 + 4(-3) \\ &= 9 - 12 \\ &= -3 \end{aligned}$$

Function Notation

Example 2

Let $f(x) = 2x + 3$ and $g(x) = x^2 + 4x$.

b. For what value(s) of x does $f(x) = -8$?

Solution

We are looking for the value of x which, when substituted into the function, produces an output value of -8.

Since $f(x) = -8$,

$$\begin{aligned} 2x + 3 &= -8 \\ 2x &= -11 \\ x &= -\frac{11}{2} \end{aligned}$$

Therefore, when $x = -\frac{11}{2}$, $f\left(-\frac{11}{2}\right) = -8$.

Function Notation

Example 2

Let $f(x) = 2x + 3$ and $g(x) = x^2 + 4x$.
c. For what value(s) of x does $f(x) = g(x)$?

Solution

$$\begin{aligned} f(x) &= g(x) \\ 2x + 3 &= x^2 + 4x \\ 0 &= x^2 + 2x - 3 \\ 0 &= (x + 3)(x - 1) \end{aligned}$$

Therefore, $x = -3$ or $x = 1$.
It follows that $f(-3) = g(-3) = -3$ and $f(1) = g(1) = 5$.

Function Notation

Example 2

Let $f(x) = 2x + 3$ and $g(x) = x^2 + 4x$.
d. Evaluate $f(g(5))$.

Solution

We always evaluate the work inside brackets first.
So in this case, we evaluate $g(5)$ by substituting 5 for x into $g(x)$, and then substitute that result for x into $f(x)$.
Since $g(x) = x^2 + 4x$,

$$\begin{aligned} g(5) &= (5)^2 + 4(5) \\ &= 25 + 20 \\ &= 45 \end{aligned}$$

Now,

$$\begin{aligned} f(g(5)) &= f(45) \\ &= 2(45) + 3 \\ &= 93 \end{aligned}$$

Therefore, $f(g(5)) = 93$.

Function Notation

Example 2

Let $f(x) = 2x + 3$ and $g(x) = x^2 + 4x$.
e. Simplify $f(g(x))$.

Solution

Since $f(x) = 2x + 3$ and $g(x) = x^2 + 4x$,

$$\begin{aligned} f(g(x)) &= f(x^2 + 4x) \\ &= 2(x^2 + 4x) + 3 \\ &= 2x^2 + 8x + 3 \end{aligned}$$

f. Simplify $g(f(x))$.

Solution

Since $g(x) = x^2 + 4x$ and $f(x) = 2x + 3$,

$$\begin{aligned} g(f(x)) &= g(2x + 3) \\ &= (2x + 3)^2 + 4(2x + 3) \\ &= 4x^2 + 12x + 9 + 8x + 12 \\ &= 4x^2 + 20x + 21 \end{aligned}$$

Generally speaking, $f(g(x)) \neq g(f(x))$.
However, there are functions $f(x)$ and $g(x)$ for which $f(g(x)) = g(f(x))$.
These will be investigated in future modules.

Composite Functions

$$f(x) = 2x + 3$$
$$g(x) = x^2 + 4x$$
$$f(g(x)) = 2x^2 + 8x + 3 \text{ and } g(f(x)) = 4x^2 + 20x + 21$$

$f(g(x))$ and $g(f(x))$ are examples of composite functions.

> **Composition of functions** is the process of combining two or more functions where one function is performed first and the result is substituted in place of x into the next function, and so on. When we substitute one function into another function, we create a **composite function**.

$f(g(x))$ is read "f of g of x". It is also written $(f \circ g)(x)$.
The idea of composite functions will be developed in other modules of this course.
We will answer questions like, "When is it possible to compose two functions?", and, "Do two functions exist such that $f(g(x)) = g(f(x))$?"

Summary

The concepts introduced in this module may not have been totally new, but they provide some of the basic building blocks for further study of functions.

- We introduced relations and functions.
- We represented relations and functions using set notation, mapping diagrams, and graphs.
- We introduced the vertical line test.
- We introduced domain and range.
- We introduced function notation.
- We introduced composite functions.