Products and Quotients of Functions

Introduction

In this module, we will investigate the behaviour and discuss the properties of functions formed by multiplying or dividing functions.

If f and g are two functions, the **product function**, denoted by $f \cdot g$ (or $f \times g$), is defined by

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

Similarly, the **quotient function**, denoted by $\dfrac{f}{g}$, is defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The domain of the product or quotient function is the set of all real numbers, x, in the domain of both f and g, with possible added restrictions on x for $\left(\dfrac{f}{g}\right)(x)$ to ensure $g(x) \neq 0$.

Examples

Example 1

Consider the two functions $f(x) = x$ and $g(x) = x^2 - 4$.

a. Determine the equations of the functions $(f \cdot g)(x)$ and $\left(\dfrac{f}{g}\right)(x)$ and identify the domain of each.

Solution

$$\begin{aligned}(f \cdot g)(x) &= f(x) \cdot g(x)\\ &= x(x^2 - 4)\\ &= x^3 - 4x\end{aligned}$$

$f \cdot g$ is a polynomial function, as are f and g, so the domain is $\{x \mid x \in \mathbb{R}\}$.

$$\begin{aligned}\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}\\ &= \frac{x}{x^2 - 4}\end{aligned}$$

$\dfrac{f}{g}$ is a rational function which is undefined when $x^2 - 4 = 0$.

Thus, the domain of $\dfrac{f}{g}$ is $\{x \mid x \neq \pm 2, x \in \mathbb{R}\}$.

Examples

Example 1

Consider the two functions $f(x) = x$ and $g(x) = x^2 - 4$.

b. Identify the coordinates of the point on each function f, g, $f \cdot g$, and $\dfrac{f}{g}$ when $x = -1$. What do you notice?

Solution

$$\begin{aligned} f(x) &= x \\ f(-1) &= -1 \end{aligned}$$

$$\begin{aligned} g(x) &= x^2 - 4 \\ g(-1) &= (-1)^2 - 4 \\ &= -3 \end{aligned}$$

$$\begin{aligned} (f \cdot g)(x) &= x^3 - 4x \\ (f \cdot g)(-1) &= (-1)^3 - 4(-1) \\ &= 3 \end{aligned}$$

$$\begin{aligned} \left(\frac{f}{g}\right)(x) &= \frac{x}{x^2 - 4} \\ \left(\frac{f}{g}\right)(-1) &= \frac{-1}{(-1)^2 - 4} \\ &= \frac{1}{3} \end{aligned}$$

From these calculations, we know that $(-1, -1)$ is a point on f and $(-1, -3)$ is a point on g.

The corresponding point on $f \cdot g$ is $(-1, 3)$. The y-coordinate of this point, 3, can be obtained by multiplying the corresponding y-coordinates of f and g; i.e., -1 and -3.

In general, for any given value of x in the domain of f and g, the corresponding y-coordinate on the product function, $(f \cdot g)(x)$, can be obtained by multiplying the corresponding y-coordinates of $f(x)$ and $g(x)$.

Examples

Example 1

Consider the two functions $f(x) = x$ and $g(x) = x^2 - 4$.

b. Identify the coordinates of the point on each function f, g, $f \cdot g$, and $\dfrac{f}{g}$ when $x = -1$. What do you notice?

Solution

$$\begin{aligned} f(x) &= x \\ f(-1) &= -1 \end{aligned}$$

$$\begin{aligned} g(x) &= x^2 - 4 \\ g(-1) &= (-1)^2 - 4 \\ &= -3 \end{aligned}$$

$$\begin{aligned} (f \cdot g)(x) &= x^3 - 4x \\ (f \cdot g)(-1) &= (-1)^3 - 4(-1) \\ &= 3 \end{aligned}$$

$$\begin{aligned} \left(\frac{f}{g}\right)(x) &= \frac{x}{x^2 - 4} \\ \left(\frac{f}{g}\right)(-1) &= \frac{-1}{(-1)^2 - 4} \\ &= \frac{1}{3} \end{aligned}$$

Similarly, the corresponding point on $\left(\dfrac{f}{g}\right)(x)$ is $\left(-1, \dfrac{1}{3}\right)$. The y-coordinate of this point, $\dfrac{1}{3}$, can be obtained by dividing the y-coordinate of f by the corresponding y-coordinate of g. That is, $\dfrac{-1}{-3}$, which is $\dfrac{1}{3}$.

In general, for any given value of x in the domain of f and g, the corresponding y-coordinate on the quotient function , $\left(\dfrac{f}{g}\right)(x)$, can be obtained by dividing the y-coordinate of $f(x)$ by the corresponding y-coordinate of $g(x)$, where $g(x) \neq 0$.

Examples

Example 1

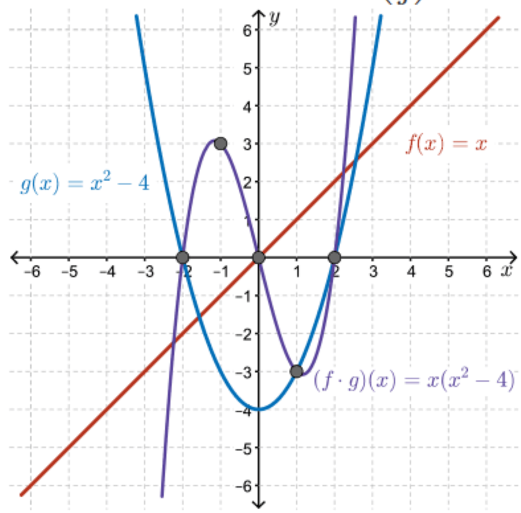
Consider the two functions $f(x) = x$ and $g(x) = x^2 - 4$.

c. Using the graphs of $f(x) = x$ and $g(x) = x^2 - 4$, sketch the graphs of $y = (f \cdot g)(x)$ and $y = \left(\frac{f}{g}\right)(x)$.

Solution

First, graph $f(x) = x$ and $g(x) = x^2 - 4$ on the same axes. Note that the domain of both functions is the set of all real numbers. By multiplying the corresponding y-coordinates of points on f and g, we can determine points on the graph of $f \cdot g$.

x	$f(x)$	$g(x)$	$(f \cdot g)(x)$
-3	-3	5	$-3 \times 5 = -15$
-2	-2	0	0
-1	-1	-3	3
0	0	1	0
1	1	-3	-3
2	2	0	0
3	3	5	15



Examples

Example 1

Consider the two functions $f(x) = x$ and $g(x) = x^2 - 4$.

c. Using the graphs of $f(x) = x$ and $g(x) = x^2 - 4$, sketch the graphs of $y = (f \cdot g)(x)$ and $y = \left(\frac{f}{g}\right)(x)$.

Solution

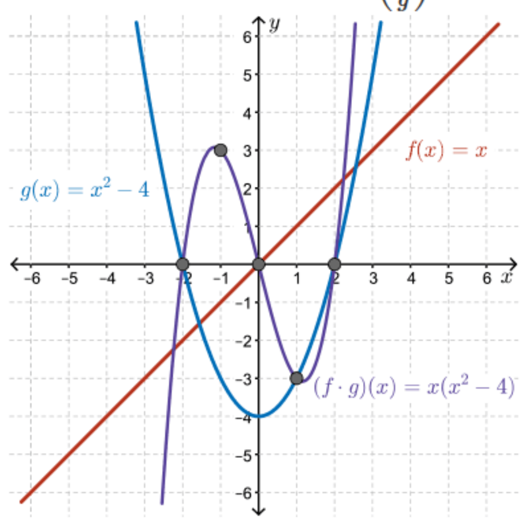
The product function, $(f \cdot g)(x) = x^3 - 4x$, is a cubic polynomial function.

The leading coefficient is positive, so $y \to -\infty$ as $x \to -\infty$, and $y \to \infty$ as $x \to \infty$.

From the factored form of the equation, we can verify that the zeros are -2, 0, and 2.

$$\begin{aligned}(f \cdot g)(x) &= x^3 - 4x \\ 0 &= x^3 - 4x \\ 0 &= x(x^2 - 4) \\ 0 &= x(x-2)(x+2) \\ 0 &= -2, 0, 2\end{aligned}$$

All the zeros are of order (multiplicity) 1, so the graph will pass directly through the x-axis at each zero.



Examples

Example 1

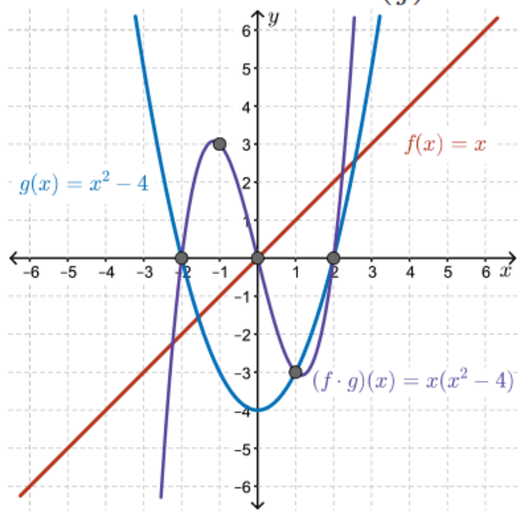
Consider the two functions $f(x) = x$ and $g(x) = x^2 - 4$.

c. Using the graphs of $f(x) = x$ and $g(x) = x^2 - 4$, sketch the graphs of $y = (f \cdot g)(x)$ and $y = \left(\frac{f}{g}\right)(x)$.

Solution

Some general observations:

- The zeros of $f \cdot g$ consist of all the zeros of the two functions f and g.
- When the graphs of f and g are both above the x-axis, or both below the x-axis, then the graph of $f \cdot g$ is above the x-axis.
- When the graphs of f and g lie on opposite sides of the x-axis, then the graph of $f \cdot g$ is below the x-axis.
- When $f(x) = 1$, $(f \cdot g)(x)$ intersects $g(x)$ since
  $$(f \cdot g)(x) = f(x) \cdot g(x) = (1)g(x) = g(x)$$
- When $g(x) = 1$, $(f \cdot g)(x)$ intersects $f(x)$ since
  $$(f \cdot g)(x) = f(x) \cdot g(x) = f(x)(1) = f(x)$$
- f is an odd function, and g is an even function. The product $f \cdot g$ is an odd function.



Examples

Example 1

Consider the two functions $f(x) = x$ and $g(x) = x^2 - 4$.

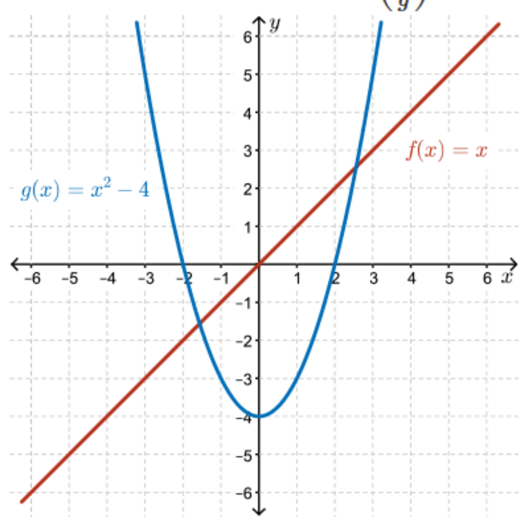
c. Using the graphs of $f(x) = x$ and $g(x) = x^2 - 4$, sketch the graphs of $y = (f \cdot g)(x)$ and $y = \left(\frac{f}{g}\right)(x)$.

Solution

To determine the graph of $\left(\frac{f}{g}\right)(x) = \frac{x}{x^2 - 4}$, we start with the graph $f(x) = x$ and $g(x) = x^2 - 4$.

The domain of $\frac{f}{g}$ is $\{x \mid x \neq \pm 2, x \in \mathbb{R}\}$ since $g(x) \neq 0$.

By dividing the y-coordinates of f by the corresponding y-coordinates of g, we can determine points on the graph of $\frac{f}{g}$.



Examples

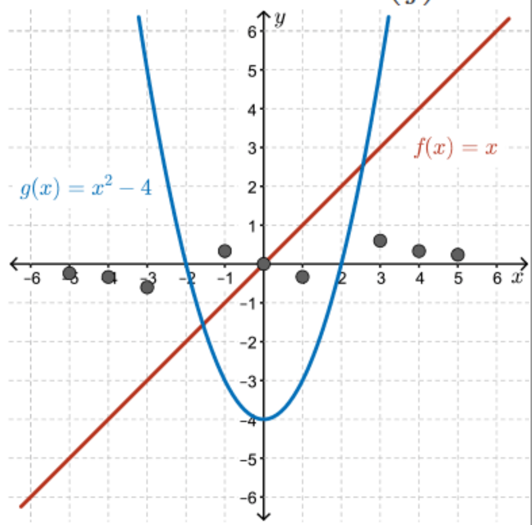
Example 1

Consider the two functions $f(x) = x$ and $g(x) = x^2 - 4$.

c. Using the graphs of $f(x) = x$ and $g(x) = x^2 - 4$, sketch the graphs of $y = (f \cdot g)(x)$ and $y = \left(\frac{f}{g}\right)(x)$.

Solution

x	$f(x)$	$g(x)$	$\left(\frac{f}{g}\right)(x)$
-5	-5	21	$-\frac{5}{21}$
-4	-4	12	$-\frac{1}{3}$
-3	-3	5	$-\frac{3}{5}$
-2	-2	0	undefined
-1	-1	-3	$\frac{1}{3}$
0	0	1	0
1	1	-3	$-\frac{1}{3}$
2	2	0	undefined
3	3	5	$\frac{3}{5}$
4	4	12	$\frac{1}{3}$
5	5	21	$\frac{5}{21}$



Examples

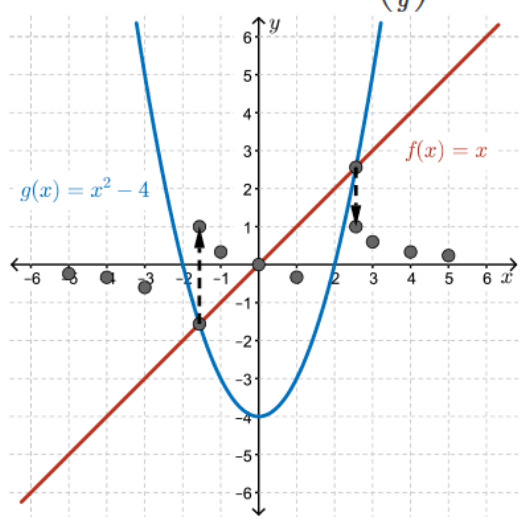
Example 1

Consider the two functions $f(x) = x$ and $g(x) = x^2 - 4$.

c. Using the graphs of $f(x) = x$ and $g(x) = x^2 - 4$, sketch the graphs of $y = (f \cdot g)(x)$ and $y = \left(\frac{f}{g}\right)(x)$.

Solution

Additional point(s) on $\frac{f}{g}$ can be identified by considering the points of intersection of f and g. At these points, $f(x) = g(x)$, so $\left(\frac{f}{g}\right)(x) = 1$.



Examples

Example 1

Consider the two functions $f(x) = x$ and $g(x) = x^2 - 4$.

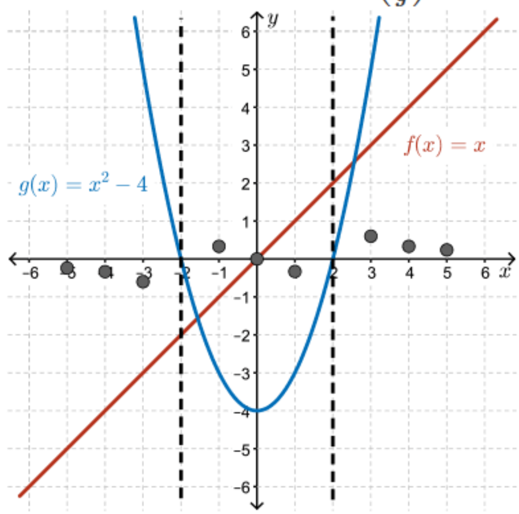
c. Using the graphs of $f(x) = x$ and $g(x) = x^2 - 4$, sketch the graphs of $y = (f \cdot g)(x)$ and $y = \left(\dfrac{f}{g}\right)(x)$.

Solution

At $x = \pm 2$, $g(x) = 0$, but $f(x) \neq 0$, so $\left(\dfrac{f}{g}\right)(x)$ has vertical asymptotes $x = 2$ and $x = -2$.

Since the degree of $g(x)$ is greater than the degree of $f(x)$, then $\left(\dfrac{f}{g}\right)(x) \to 0$ as $x \to \pm\infty$.

Thus, $y = 0$ is a horizontal asymptote.



Examples

Example 1

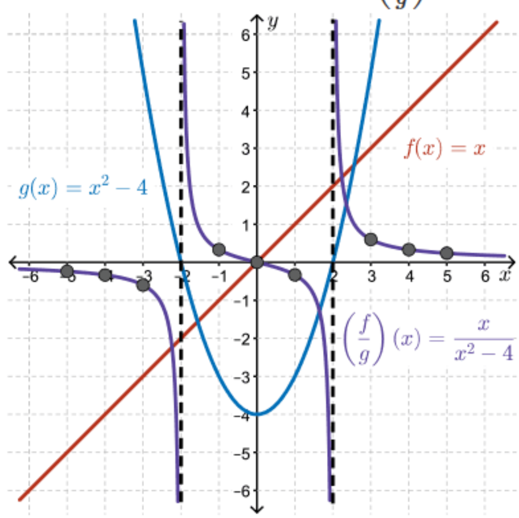
Consider the two functions $f(x) = x$ and $g(x) = x^2 - 4$.

c. Using the graphs of $f(x) = x$ and $g(x) = x^2 - 4$, sketch the graphs of $y = (f \cdot g)(x)$ and $y = \left(\dfrac{f}{g}\right)(x)$.

Solution

When $x < -2$, $f < 0$ (below the x-axis) and $g > 0$ (above the x-axis), so $\dfrac{f}{g} < 0$ and its graph will approach the x-axis from below as $x \to -\infty$ and approach $-\infty$ as $x \to -2^-$. When $-2 < x < 0$, both $f < 0$ and $g < 0$, so $\dfrac{f}{g} > 0$ and the graph approaches $+\infty$ as $x \to -2^+$.

In a similar way, we can determine the behaviour of the graph to the right of the y-axis. Notice that $\left(\dfrac{f}{g}\right)(x)$ has odd symmetry.



Examples

Example 2

Consider the two functions $f(x) = \log_2(x)$ and $g(x) = \sqrt{8-x}$.
a. Determine the equation of $p(x)$ where $p(x) = (f \cdot g)(x)$, and state its domain.

Solution

Determining the equation for $p(x)$, we have

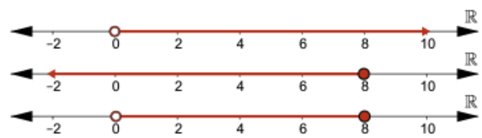
$$\begin{aligned} p(x) &= (f \cdot g)(x) \\ &= f(x) \cdot g(x) \\ &= \log_2(x) \cdot \left(\sqrt{8-x}\right) \end{aligned}$$

For the purpose of clarity, we will express $p(x)$ as $p(x) = \sqrt{8-x}\,\log_2(x)$.

The domain of $f(x) = \log_2(x)$ is $\{x \mid x > 0, x \in \mathbb{R}\}$.

The domain of $g(x) = \sqrt{8-x}$ is $\{x \mid x \le 8, x \in \mathbb{R}\}$.

The domain of $p(x) = \sqrt{8-x} \cdot \log_2(x)$ is the set of all real numbers such that $x > 0$ and $x \le 8$.



That is, $\{x \mid x > 0, x \in \mathbb{R}\} \cap \{x \mid x \le 8, x \in \mathbb{R}\}$.
Therefore, the domain of $p(x)$ is $\{x \mid 0 < x \le 8, x \in \mathbb{R}\}$, or, using interval notation, $x \in (0, 8], x \in \mathbb{R}$.

Examples

Example 2

Consider the two functions $f(x) = \log_2(x)$ and $g(x) = \sqrt{8-x}$.
b. Using the graphs of f and g, predict the behaviour of the graph of the product function p.

Solution

To graph $p(x) = \sqrt{8-x}\,\log_2(x)$, we start with the graph of $f(x) = \log_2(x)$ and $g(x) = \sqrt{8-x}$.

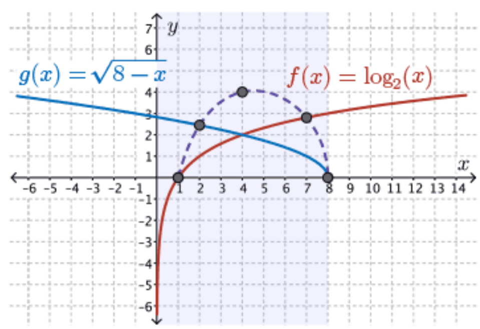
We can obtain points on $p(x) = \sqrt{8-x}\,\log_2(x)$ by multiplying the corresponding y-coordinates of $f(x)$ and $g(x)$ for values of x in the domain of both f and g.

That is, for real values of x, $0 < x \le 8$.

Since $p(x) = f(x) \cdot g(x)$, then the zeros of $p(x)$ occur when $f(x) = 0$ or $g(x) = 0$.

Thus, the zeros of $p(x)$ are at $x = 1$ and $x = 8$.

Since $f(2) = 1$, then
$p(2) = f(2) \cdot g(2) = 1 \cdot g(2) = g(2)$.



Similarly, $g(7) = 1$, so $p(7) = f(7)$.

The point $(4, 2)$, is on both $f(x)$ and $g(x)$, so $(4, (2)(2)) = (4, 4)$ is a point on $p(x)$.

Examples

Example 2

Consider the two functions $f(x) = \log_2(x)$ and $g(x) = \sqrt{8-x}$.

b. Using the graphs of f and g, predict the behaviour of the graph of the product function p.

Solution

To graph $p(x) = \sqrt{8-x}\log_2(x)$, we start with the graph of $f(x) = \log_2(x)$ and $g(x) = \sqrt{8-x}$.

To complete the sketch, we must consider the behaviour of the function as x approaches 0 from the right side.

As $x \to 0^+$,

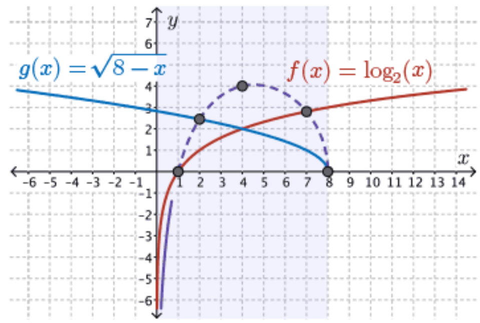
$$f(x) \to -\infty$$

and

$$g(x) \to \sqrt{8}$$

so

$$p(x) \to -\infty$$



Examples

Example 2

Consider the two functions $f(x) = \log_2(x)$ and $g(x) = \sqrt{8-x}$.

b. Using the graphs of f and g, predict the behaviour of the graph of the product function p.

Solution

To graph $p(x) = \sqrt{8-x}\log_2(x)$, we start with the graph of $f(x) = \log_2(x)$ and $g(x) = \sqrt{8-x}$.

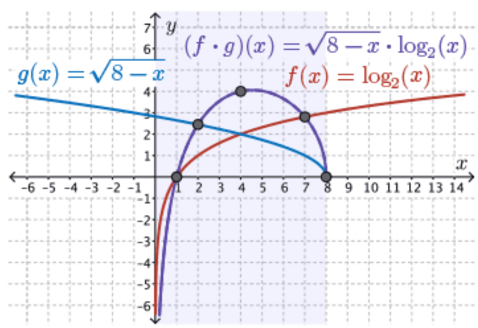
Knowing the behaviour of the component functions, $f(x) = \log_2(x)$ and $g(x) = \sqrt{8-x}$, was helpful in sketching the graph of the more complicated product function,

$$p(x) = \sqrt{8-x}\log_2(x)$$

Analyzing the behaviour of the corresponding quotient function,

$$\left(\frac{f}{g}\right)(x) = \frac{\log_2(x)}{\sqrt{8-x}}$$

has been left as an exercise problem.



Investigation

Take some time now to investigate the behaviour of a variety of different product and quotient functions using the worksheet provided.

Questions to consider:

- Is the graph of $(f \cdot g)(x)$ the same as the graph of $(g \cdot f)(x)$?
- How does the graph of $\left(\dfrac{g}{f}\right)(x)$ compare to the graph of $\left(\dfrac{f}{g}\right)(x)$? Are they the same? If not, is there a relationship between them?
- How does the symmetry of each function, f and g, influence the symmetry of $f \cdot g$ or $\dfrac{f}{g}$?
- What happens when a sinusoidal function is combined with a non-periodic function? What happens when a sinusoidal function is combined with another sinusoidal function?

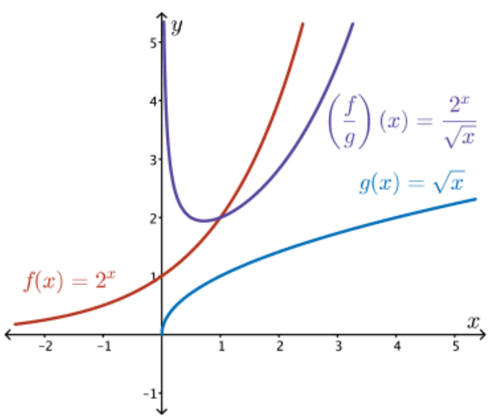
Observations

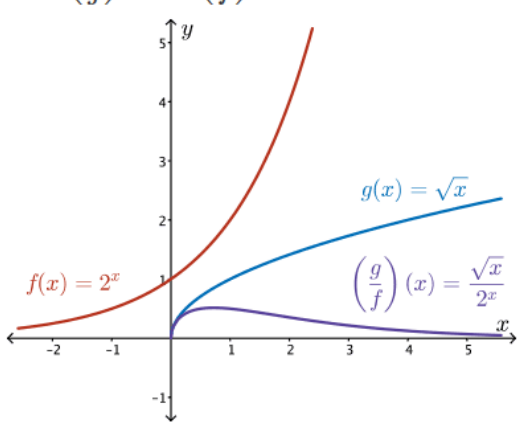
The graph of $(f \cdot g)(x)$ was the same as the graph of $(g \cdot f)(x)$.
The multiplication of functions is commutative; that is,

$$f(x)g(x) = g(x)f(x)$$

Thus, $(f \cdot g)(x) = (g \cdot f)(x)$.

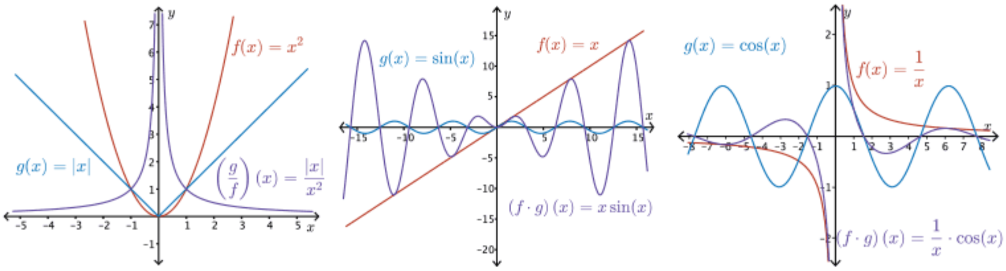
There does not appear to be a direct relationship between the graph $\left(\dfrac{f}{g}\right)(x)$ and $\left(\dfrac{g}{f}\right)(x)$.





Observations

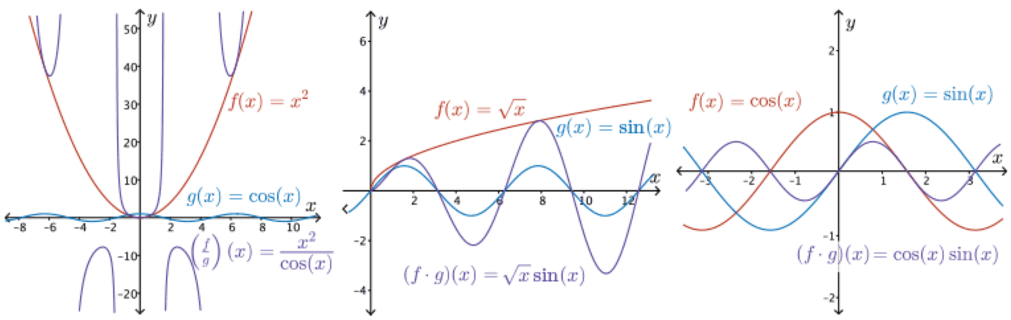
The product or quotient of two even functions was an even function.
The product or quotient of two odd functions was an even function.
When one of the component functions was even and the other was odd, the product or quotient function was odd.



These statements hold true in general as long as f and g are non-zero functions. Try to prove these results algebraically.

Observations

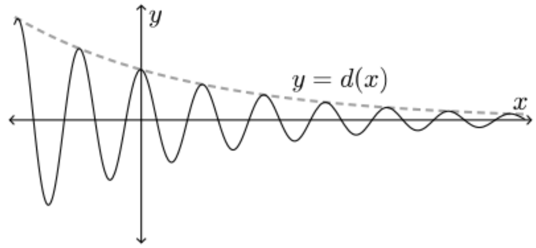
When exactly one of f or g was a sinusoidal function, the product and quotient functions were not necessarily periodic (they did not repeat themselves over a specific interval), but the behaviour of their graphs might suggest that one of the component functions is periodic.
When both f and g were sinusoidal, the product and quotient functions appeared to be periodic.



Examples

Example 3

Consider the function, $y = d(x)$, shown here.
The graph of this function models damped sinusoidal motion, typical of a free-swinging pendulum.
Identify two component functions, f and g, whose product or quotient function would model behaviour similar to $y = d(x)$? Provide reasoning for your choice of functions. Use graphing technology to verify your choice.



Solution

One of the component functions is sinusoidal.
Since a local maximum occurs at $x = 0$, it is easiest to use the cosine function.
Recall, for $y = a\cos(x)$, the amplitude is $|a|$.
We seek a product function, $y = f(x)\cos(x)$, such that the function $f(x)$ causes a "varying amplitude" for the curve, so the local maximum and minimum values will be determined by the value of $f(x)$ at specific values of x.
For $y = d(x)$, the height of each successive peak gradually decreases, approaching 0 as $x \to \infty$.
Thus, $f(x)$ must be a decreasing function that approaches a horizontal asymptote of $y = 0$ as x increases in value.
Also, note that $d(x)$ has neither even nor odd symmetry.
The function $y = \cos(x)$ is even so $f(x)$ must be neither even nor odd.

Examples

Example 3

Identify two component functions, f and g, whose product or quotient function would model behaviour similar to $y = d(x)$? Provide reasoning for your choice of functions. Use graphing technology to verify your choice.

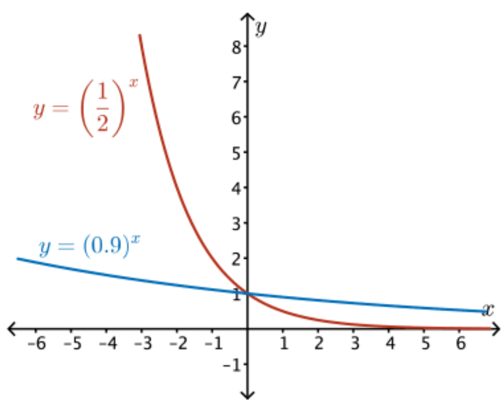
Solution

Of the various functions we have studied in this course, we know that $y = a^x, a \neq 0$, is neither an even nor an odd function, and it is an increasing function over its domain when $a > 1$ and a decreasing function when $0 < a < 1$.

For example, $y = \left(\frac{1}{2}\right)^x$ is a decreasing function that approaches a horizontal asymptote of $y = 0$ as $x \to \infty$.

However, this function decreases fairly quickly.

A more gradually decreasing exponential function will have a base slightly less than 1, such as $y = (0.9)^x$.



Examples

Example 3

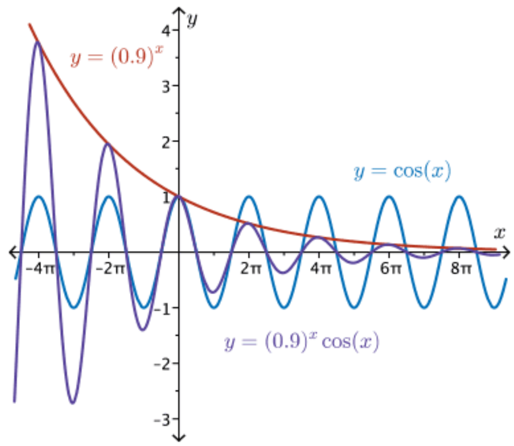
Identify two component functions, f and g, whose product or quotient function would model behaviour similar to $y = d(x)$? Provide reasoning for your choice of functions. Use graphing technology to verify your choice.

Solution

Does $y = (0.9)^x \cos(x)$ model behaviour similar to $y = d(x)$?

We can obtain the graph of $y = (0.9)^x \cos(x)$ using graphing technology.

This function models behaviour similar to that of $y = d(x)$. The rate at which the sinusoidal function is dampened can be accomplished by varying the base of the exponential function.



Examples

Example 3

Identify two component functions, f and g, whose product or quotient function would model behaviour similar to $y = d(x)$? Provide reasoning for your choice of functions. Use graphing technology to verify your choice.

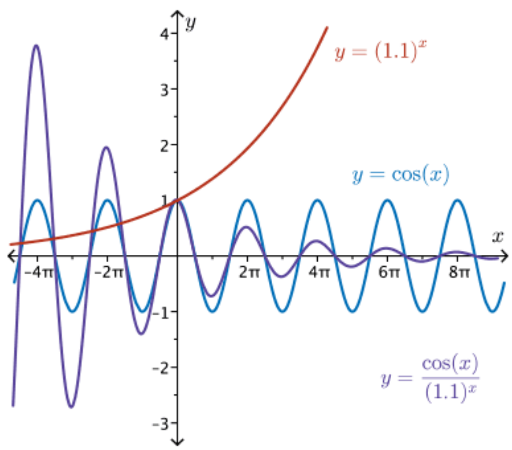
Solution

We should also note that this behaviour can be obtained by the quotient of a sinusoidal function and an exponential function, $y = a^x$, where $a > 1$.

For example, the function $y = \dfrac{\cos(x)}{(1.1)^x}$ will model behaviour similar to $y = d(x)$.

This curve can also be defined by the product function $y = (1.1)^{-x} \cos(x)$.

The most common form of sinusoidal damping is exponential damping, where the amplitude of successive peaks undergoes exponential decay.



Summary

Product and quotients of functions can be used to model many different, complicated situations.
Understanding the behaviour of the component functions helps in determining the properties and sketching the graphs of these, more complex, combined functions.

Summary

Two functions can be combined using multiplication or division to create a new (and often more complex) function.

- The product of two functions f and g is a function defined by $(f \cdot g)(x) = f(x) \cdot g(x)$.
- The quotient of two functions f and g is a function defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ where $g(x) \neq 0$.
- The domain of the product or quotient function is the set of all real numbers in the domain of both f and g, with an added restriction of $g(x) \neq 0$ for $\frac{f}{g}$.
- The graph of $f \cdot g$ can be obtained from the graphs of f and g by multiplying corresponding y-coordinates.
- The graph of $\frac{f}{g}$ can be obtained from the graphs of f and g by dividing the y-coordinates of f by the corresponding y-coordinates of g (as long as these y-values are non-zero).
- The behaviour of the graph of each (non-zero) function f and g will, in some way, influence the behaviour of the graphs of $f \cdot g$ and $\frac{f}{g}$.
- When both f and g are positive (above the x-axis), or both negative (below the x-axis), then $f \cdot g$ is positive. When one of f or g is positive and the other is negative, then $f \cdot g$ is negative. The zeros of $f \cdot g$ occur when $f(x) = 0$ or $g(x) = 0$. The zeros of $\frac{f}{g}$ occur when $f(x) = 0$ and $g(x) \neq 0$.
- For non-zero functions f and g, if both f and g are even functions, or both f and g are odd functions, then $f \cdot g$ and $\frac{f}{g}$ are even functions. If one of f or g is an odd function and the other is an even function, then $f \cdot g$ and $\frac{f}{g}$ are odd functions.