Sums and Differences of Functions

Introduction

Within this course of study, we have examined the behaviour and discussed the properties of a variety of functions: absolute value and square root functions, polynomial functions, rational functions, exponential and logarithmic functions, and trigonometric functions.

In This Unit

- We will look at ways of combining these functions to produce new functions that model more complex relationships.

In This Module

- We will focus on functions formed by the addition or subtraction of functions. We will analyze these functions numerically, algebraically, and graphically.
- We will investigate the behaviour of the graphs of these combined functions and discuss factors that affect their properties.
- We will examine a real world application requiring a difference of two functions.

Introduction

The sum and difference of two functions are defined as follows.

If f and g are two functions, the **sum function**, denoted by $f+g$, is defined by

$$(f+g)(x) = f(x) + g(x)$$

Similarly, the **difference function**, denoted by $f-g$, is defined by

$$(f-g)(x) = f(x) - g(x)$$

The sum or difference of two functions is a function whose domain is the set of all real numbers that are in the domain of both f and g.

Examples

Example 1

Consider the two functions $f(x) = \frac{1}{2}x$ and $g(x) = \sqrt{x+4}$.

a. Determine $(f+g)(x)$ and $(f-g)(x)$.

Solution

Since $(f+g)(x) = f(x) + g(x)$, then

$$(f+g)(x) = \frac{1}{2}x + \sqrt{x+4}$$

Similarly, $(f-g)(x) = f(x) - g(x)$. So,

$$(f-g)(x) = \frac{1}{2}x - \sqrt{x+4}$$

Examples

Example 1

Consider the two functions $f(x) = \frac{1}{2}x$ and $g(x) = \sqrt{x+4}$.

b. Find $f(12)$, $g(12)$, $(f+g)(12)$, and $(f-g)(12)$.

Solution

Using the equations for f and g and the results from part a),

$$\begin{aligned} f(x) &= \frac{1}{2}x \\ f(12) &= \frac{1}{2}(12) \\ &= 6 \end{aligned}$$

$$\begin{aligned} (f+g)(x) &= \frac{1}{2}x + \sqrt{x+4} \\ (f+g)(12) &= \frac{1}{2}(12) + \sqrt{12+4} \\ &= 6+4 \\ &= 10 \end{aligned}$$

$$\begin{aligned} g(x) &= \sqrt{x+4} \\ g(12) &= \sqrt{12+4} \\ &= 4 \end{aligned}$$

$$\begin{aligned} (f-g)(x) &= \frac{1}{2}x - \sqrt{x+4} \\ (f-g)(12) &= \frac{1}{2}(12) - \sqrt{12+4} \\ &= 6-4 \\ &= 2 \end{aligned}$$

Examples

Example 1

Consider the two functions $f(x) = \frac{1}{2}x$ and $g(x) = \sqrt{x+4}$.

c. Find $(f+g)(-6)$.

Solution

Since $(f+g)(x) = \frac{1}{2}x + \sqrt{x+4}$, then

$$\begin{aligned}(f+g)(-6) &= \frac{1}{2}(-6) + \sqrt{-6+4} \\ &= -3 + \sqrt{-2}\end{aligned}$$

The value of $(f+g)(-6)$ is undefined in the real numbers.

This is due to the fact that $g(x) = \sqrt{x+4}$ is undefined when $x < -4$.

The domain of $f(x) = \frac{1}{2}x$ is $\{x \mid x \in \mathbb{R}\}$. The domain of $g(x) = \sqrt{x+4}$ is $\{x \mid x \geq -4,\ x \in \mathbb{R}\}$.

The domain of $(f+g)(x)$ is the set of all real numbers that are in both the domain of $f(x)$ and $g(x)$.

That is, $\{x \mid x \in \mathbb{R}\} \cap \{x \mid x \geq -4,\ x \in \mathbb{R}\}$.

Thus, the domain of $(f+g)(x) = \frac{1}{2}x + \sqrt{x+4}$ is $\{x \mid x \geq -4,\ x \in \mathbb{R}\}$ or, using interval notation, $x \in [-4, \infty),\ x \in \mathbb{R}$.

The function $(f-g)(x) = \frac{1}{2}x - \sqrt{x+4}$ has the same domain.

Examples

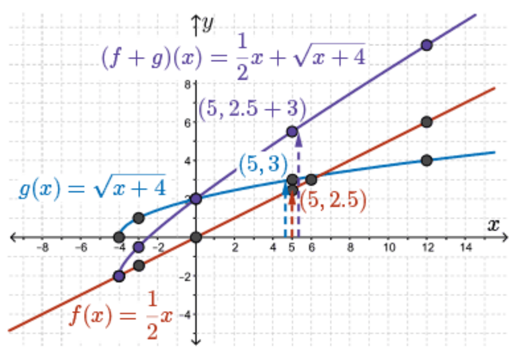
Example 1

Consider the two functions $f(x) = \frac{1}{2}x$ and $g(x) = \sqrt{x+4}$.

d. Sketch the graph of $y = (f+g)(x)$ and $y = (f-g)(x)$.

Solution

x	$f(x)$	$g(x)$	$(f+g)(x)$
-8	-4	undefined	undefined
-6	-3	undefined	undefined
-4	-2	0	$-2+0=-2$
-3	-1.5	1	$-1.5+1=-0.5$
-2	-1	$\sqrt{2}$	$-1+\sqrt{2}\approx 0.41$
0	0	2	$0+2=2$
2	1	$\sqrt{6}$	$1+\sqrt{6}\approx 3.45$
4	2	$\sqrt{8}$	$2+\sqrt{8}\approx 4.83$
5	2.5	3	$2.5+3=5.5$
6	3	$\sqrt{10}$	$3+\sqrt{10}\approx 6.16$
8	4	$\sqrt{12}$	$4+\sqrt{12}\approx 7.46$
12	6	4	$6+4=10$



First, we graph $f(x)$ and $g(x)$.

Since $(f+g)(x) = f(x) + g(x)$, the graph of $(f+g)(x) = \frac{1}{2}x + \sqrt{x+4}$ can be obtained by adding the corresponding y-coordinates for values of x in the domain of both f and g.

Examples

Example 1

Consider the two functions $f(x) = \frac{1}{2}x$ and $g(x) = \sqrt{x+4}$.

d. Sketch the graph of $y = (f+g)(x)$ and $y = (f-g)(x)$.

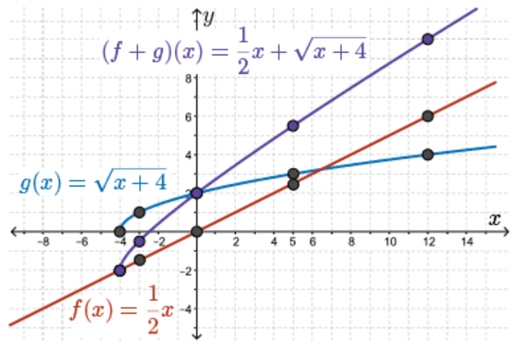
Solution

Using the graphs of $f(x) = \frac{1}{2}x$ and $g(x) = \sqrt{x+4}$, we were able to obtain a reasonable sketch of the graph of $(f+g)(x) = \frac{1}{2}x + \sqrt{x+4}$.

Knowing the behaviour of each component graph helps in predicting the behaviour of the sum function.

Note that both f and g are increasing functions and $f+g$ is also an increasing function.

Furthermore, the graph of the sum function has a shape similar to $g(x) = \sqrt{x+4}$, but its graph has also been influenced by the behaviour of $f(x) = \frac{1}{2}x$ since it increases more quickly than $g(x)$ as x increases in value.



Examples

Example 1

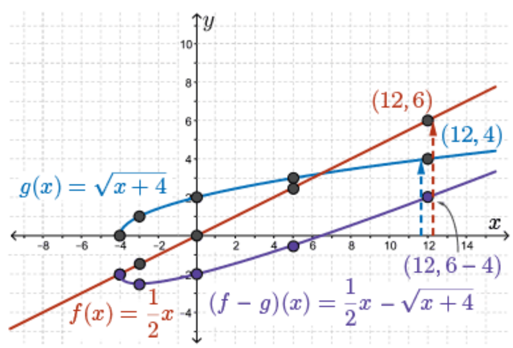
Consider the two functions $f(x) = \frac{1}{2}x$ and $g(x) = \sqrt{x+4}$.

d. Sketch the graph of $y = (f+g)(x)$ and $y = (f-g)(x)$.

Solution

The graph of $(f-g)(x) = \frac{1}{2}x - \sqrt{x+4}$ can be obtained by subtracting the y-coordinate of g from the corresponding y-coordinate of f for values of x in the domain of both f and g.

x	$f(x)$	$g(x)$	$(f-g)(x)$
-8	-4	undefined	undefined
-6	-3	undefined	undefined
-4	-2	0	$-2-0=-2$
-3	-1.5	1	$-1.5-1=-2.5$
-2	-1	$\sqrt{2}$	$-1-\sqrt{2}\approx -2.41$
0	0	2	$0-2=-2$
2	1	$\sqrt{6}$	$1-\sqrt{6}\approx -1.45$
4	2	$\sqrt{8}$	$2-\sqrt{8}\approx -0.83$
5	2.5	3	$2.5-3=-0.5$
6	3	$\sqrt{10}$	$3-\sqrt{10}\approx -0.16$
8	4	$\sqrt{12}$	$4-\sqrt{12}\approx -0.54$
12	6	4	$6-4=2$



Examples

Example 1

Consider the two functions $f(x) = \frac{1}{2}x$ and $g(x) = \sqrt{x+4}$.

d. Sketch the graph of $y = (f+g)(x)$ and $y = (f-g)(x)$.

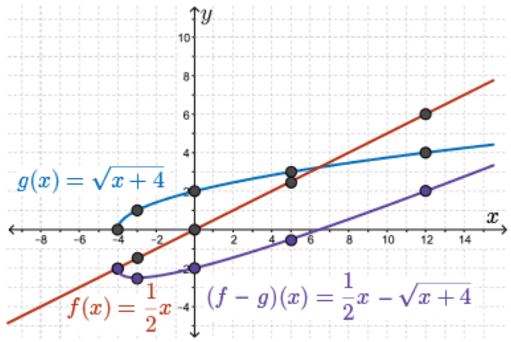
Solution

We can see that the graph of $(f-g)(x) = \frac{1}{2}x - \sqrt{x+4}$ has been influenced by the behaviour of the graphs of f and g. The graph is similar in shape to g.

In the interval where $f(x) < g(x)$, the difference function, $f-g$, is below the x-axis.

Where $f(x) > g(x)$, the difference function, $f-g$, is above the x-axis.

Wherever $f(x)$ and $g(x)$ intersect, $f-g$ has an x-intercept.



The component functions, f and g, are both increasing functions. However, $f-g$ appears to decrease from $x = -4$ to approximately $x = -3$ and then begins to increase.

Examples

Remarks

What influences these changes?

It might not be easy to answer this question now (without knowledge of calculus), but it is worth thinking about.

The following are some other questions that could be asked.

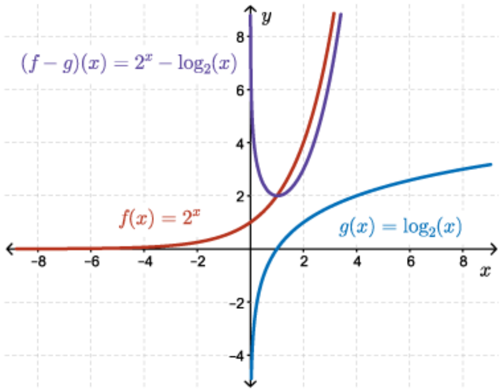
- Is the graph of $(f+g)(x)$ the same as the graph of $(g+f)(x)$?
- How does the graph of $(g-f)(x)$ compare to the graph of $(f-g)(x)$? Are they the same? If not, is there a relationship between them?
- How does the symmetry of the functions, f and g, influence the symmetry of $f+g$ or $f-g$?
- What happens when one or both of the functions are sinusoidal?

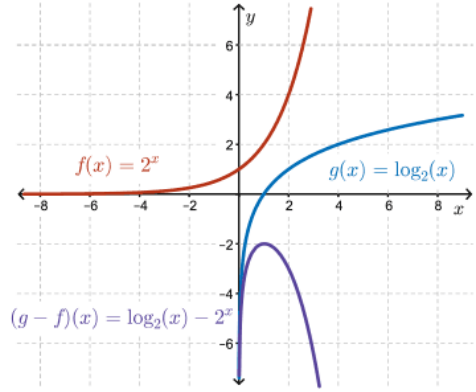
Take some time now to investigate the behaviour of a variety of sum and difference functions using the worksheet provided.

Observations

The graph of $(f+g)(x)$ was the same as the graph of $(g+f)(x)$; that is, $(f+g)(x) = (g+f)(x)$.

However, $(f-g)(x) \neq (g-f)(x)$ unless $g(x) = f(x)$.
The graph of $(g-f)(x)$ was the reflection of the graph of $(f-g)(x)$ in the x-axis.



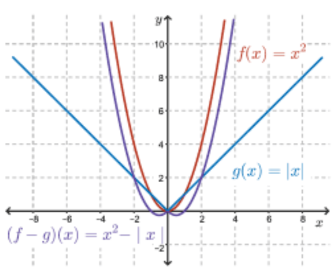


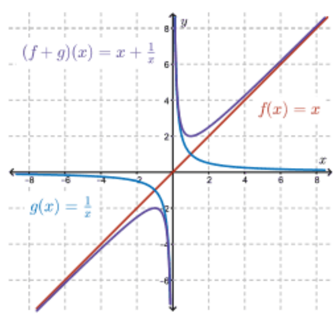
This can be shown algebraically:

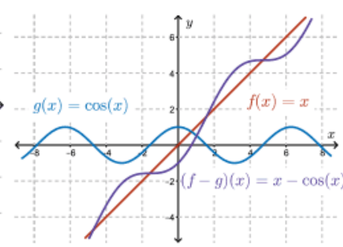
$$\begin{aligned}(g-f)(x) &= g(x) - f(x) \\ &= -[f(x) - g(x)] \\ &= -[(f-g)(x)]\end{aligned}$$

Observations

The sum or difference of two even functions was an even function.
The sum or difference of two odd functions was an odd function.
The sum or difference of an even and odd function was neither even nor odd.
These statements hold true in general as long as f and g are non-zero functions. Try to prove these results algebraically.



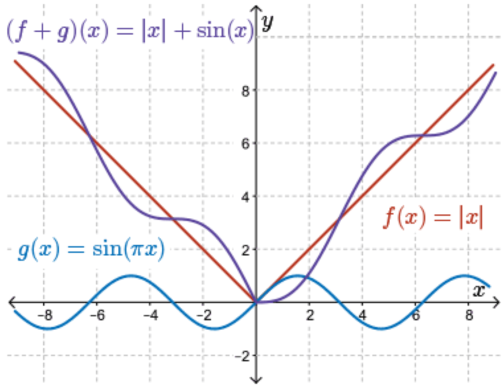


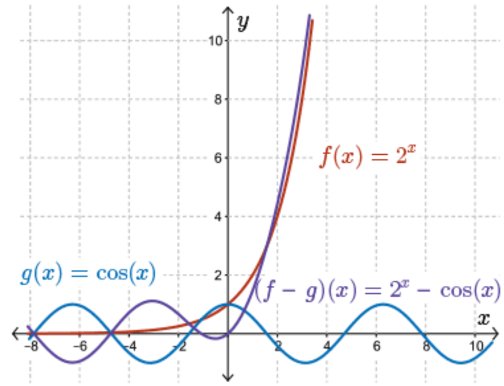


Observations

What did you notice when one or both functions were sinusoidal?

When exactly one of f or g was a sinusoidal function, the sum and difference functions were not necessarily periodic, but the behaviour of their graphs might suggest that one of the component functions is periodic.





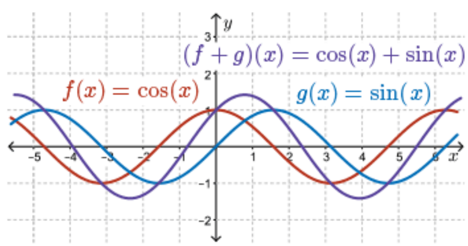
It is possible for the sum of two functions, only one of which is sinusoidal, to be periodic.

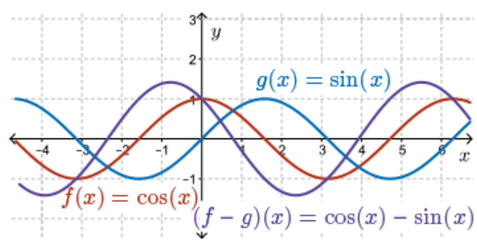
One example has a second function that is constant, such as $y = \sin(x) + 2$.

Observations

When both of the functions, f and g, were sinusodial, the sum and difference functions appeared to be periodic.

This is often, but not always, the case when working with sinusoidal functions.





Examples

Example 2

A culinary class at school runs an annual cake sale. In the past, they have sold an average of 30 cakes at $\$12$ per cake. It is predicted that for every $\$1$ decrease in price, 5 more cakes will be sold, and for every $\$1$ increase in price, 5 fewer cakes will be sold. The cost of ingredients to make each cake is approximately $\$2$. To keep money collection simple, only full $\$1$ changes (increases or decreases) in the cake price will be considered.

Based on this information and given that x represents the number of $\$1$ decreases in the selling price per cake, answer the following:

a. Determine a function R, in terms of x, to model the revenue that could be generated from the cake sale. Identify the domain of the function.
b. Determine a function C, in terms of x, to model the cost of the cakes sold at the cake sale. Identify a domain for the function.
c. Graph R and C on the same axes. Using these graphs, sketch a graph of a function that models the profit from the cake sale. Identify the domain of this new function.
d. Determine the equation of the profit function. Use this equation to verify the graph of the profit function that was produced in part c).

Examples

Example 2

Facts: In the past, 30 cakes were sold for $\$12$ per cake. For every $\$1$ decrease in price, 5 more cakes will be sold, and for every $\$1$ increase in price, 5 fewer cakes will be sold. The number of $\$1$ dollar decreases in price will be represented by x. The cost to make a cake is $\$2$.

a. Determine a function R, in terms of x, to model the revenue that could be generated from the cake sale. Identify the domain of the function.

Solution

If x represents the number of $\$1$ decreases in the selling price per cake, then

- the selling price per cake in dollars is given by $12 - 1x$, and
- the number of cakes sold is given by $30 + 5x$.

Since

$$\text{Revenue} = \text{Price per Cake} \times \text{Number of Cakes Sold}$$

the revenue R generated from selling the cakes is given by

$$R(x) = (12 - x)(30 + 5x)$$
$$R(x) = -5x^2 + 30x + 360$$

Examples

Example 2

Facts: In the past, 30 cakes were sold for $\$12$ per cake. For every $\$1$ decrease in price, 5 more cakes will be sold, and for every $\$1$ increase in price, 5 fewer cakes will be sold. The number of $\$1$ dollar decreases in price will be represented by x. The cost to make a cake is $\$2$.

a. Determine a function R, in terms of x, to model the revenue that could be generated from the cake sale. Identify the domain of the function.

Solution

$$R(x) = (12 - x)(30 + 5x)$$

To determine the domain of $R(x)$, we note that only full $\$1$ changes in the cake price will be allowed, so $x \in \mathbb{Z}$.

Furthermore, both the selling price per cake and the number of cakes sold must be greater than or equal to zero.

Price per cake:

$$\begin{aligned} 12 - x &\geq 0 \\ -x &\geq -12 \\ x &\leq 12 \end{aligned}$$

and

The number sold:

$$\begin{aligned} 30 + 5x &\geq 0 \\ 5x &\geq -30 \\ x &\geq -6 \end{aligned}$$

Thus, the domain of $R(x)$ is $\{x \mid -6 \leq x \leq 12,\ x \in \mathbb{Z}\}$.

Examples

Example 2

Facts: In the past, 30 cakes were sold for $\$12$ per cake. For every $\$1$ decrease in price, 5 more cakes will be sold, and for every $\$1$ increase in price, 5 fewer cakes will be sold. The number of $\$1$ dollar decreases in price will be represented by x. The cost to make a cake is $\$2$.

b. Determine a function C, in terms of x, to model the cost of the cakes sold at the cake sale. Identify a domain for the function.

Solution

The cost to make each cake is $\$2$. So, the cost of the cakes sold, $C(x)$, is

$$C(x) = 2(30 + 5x)$$

or

$$C(x) = 10x + 60$$

Since $C(x) \geq 0$,

$$\begin{aligned} 10x + 60 &\geq 0 \\ 10x &\geq -60 \\ x &\geq -6 \end{aligned}$$

Therefore, the domain of the cost function, defined by $C(x) = 10x + 60$, is $\{x \mid x \geq -6,\ x \in \mathbb{Z}\}$.

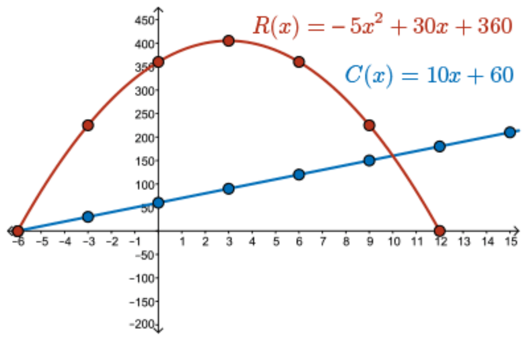
Examples

Example 2

c. Graph R and C on the same axes. Using these graphs, sketch a graph of a function that models the profit from the cake sale. Identify the domain of this new function.

Solution

x	$R(x)$	$C(x)$	$(R-C)(x)$
-6	0	0	
-3	225	30	
0	350	60	
3	405	90	
6	360	120	
9	225	150	
12	0	180	
15		210	



Using a table of values where x is in the domain of the function, we produce the graph of $R(x) = (12 - x)(30 + 5x)$ and $C(x) = 2(30 + 5x)$. Since x is an integer, the graphs of $R(x)$ and $C(x)$ consist of a set of non-connected points. We will connect these points to illustrate the relationship.

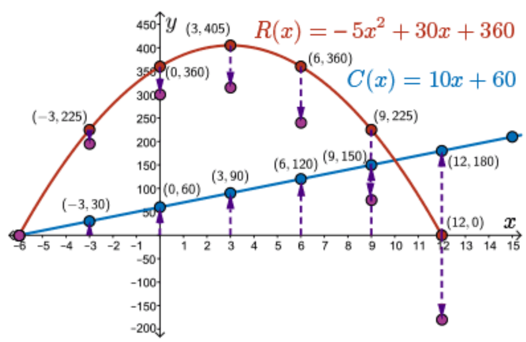
Examples

Example 2

c. Graph R and C on the same axes. Using these graphs, sketch a graph of a function that models the profit from the cake sale. Identify the domain of this new function.

Solution

x	$R(x)$	$C(x)$	$(R-C)(x)$
-6	0	0	0
-3	225	30	195
0	350	60	300
3	405	90	315
6	360	120	240
9	225	150	75
12	0	180	-180
15		210	



Now, profit is monetary gain; it is the difference between revenue and cost. Thus, profit, $P(x)$, is given by $R(x) - C(x)$. To obtain the graph of $P(x)$, we subtract the y-values of $C(x)$ from the corresponding y-values of $R(x)$ for values of x in the domain of both R and C.

Examples

Example 2

c. Graph R and C on the same axes. Using these graphs, sketch a graph of a function that models the profit from the cake sale. Identify the domain of this new function.

Solution

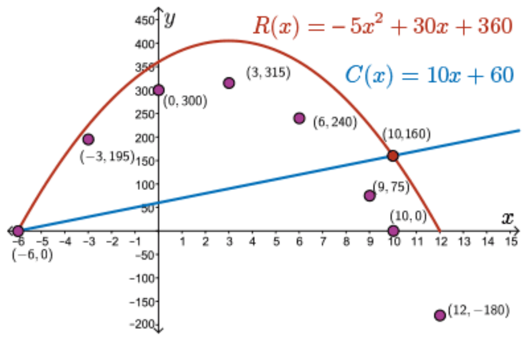
This profit function is quadratic.

To complete the graph, we should identify the zeros and the vertex.

The zeros occur when $R(x) - C(x) = 0$. That is, when $R(x) = C(x)$.

These points are called **break-even** points since the revenue generated from sales equals the cost (profit is zero).

One zero occurs at $x = -6$ and the second occurs at $x = 10$, where $R(x)$ and $C(x)$ intersect again.



$$R(10) = -5(10)^2 + 30(10) + 360 = 160$$
$$C(10) = 10(10) + 60 = 160$$
$$R(10) - C(10) = 0$$

Therefore, $P(10) = 0$.

Examples

Example 2

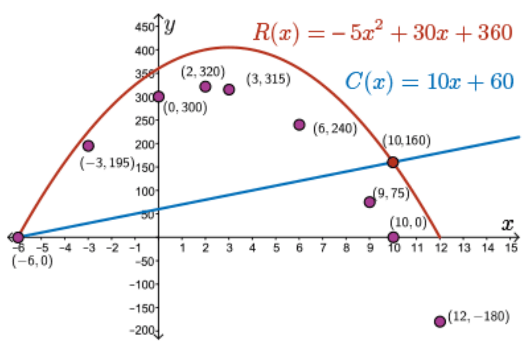
c. Graph R and C on the same axes. Using these graphs, sketch a graph of a function that models the profit from the cake sale. Identify the domain of this new function.

Solution

The vertex occurs along the axis of symmetry, midway between the two zeros.

Thus, the vertex occurs at $x = \dfrac{-6+10}{2} = 2$.

$$R(2) = -5(2)^2 + 30(2) + 360 = 400$$
$$C(2) = 10(2) + 60 = 80$$
$$R(2) - C(2) = 320$$



Therefore, $P(2) = 320$.

The vertex is $(2, 320)$.

The profit function is increasing when $-6 \le x < 2$ and decreasing when $2 < x \le 12$.

A maximum profit of $\$320$ can be achieved with two $\$1$ decreases in the selling price of the cake.

That is, when the cakes are sold for $\$10$: $\text{Selling Price per Cake} = 12 - x = 12 - 2 = 10$.

Examples

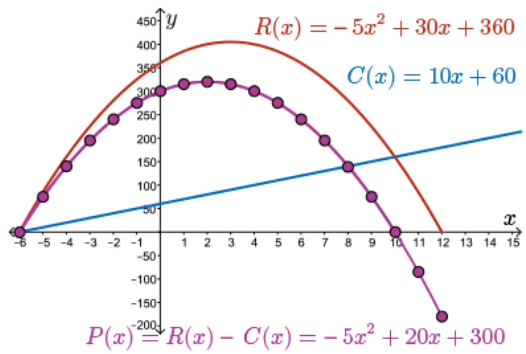
Example 2

c. Graph R and C on the same axes. Using these graphs, sketch a graph of a function that models the profit from the cake sale. Identify the domain of this new function.

Solution

Since x is an integer, the graph of $P(x)$ consists of a set of non-connected points.
We will connect the points to provide a better visual.
The domain of $P(x)$ is the set of all values in the domain of both $R(x)$ and $C(x)$.
Therefore, the domain is given by

$$\{x \mid -6 \le x \le 12,\ x \in \mathbb{Z}\}$$



Examples

Example 2

d. Determine the equation of the profit function. Use this equation to verify the graph of the profit function that was produced in part c).

Solution

Since $\textbf{Profit} = \textbf{Revenue} - \textbf{Cost}$, the equation of the profit function is given by $(R - C)(x)$.
Thus,

$$\begin{aligned} P(x) &= (R - C)(x) \\ &= (-5x^2 + 30x + 360) - (10x + 60) \\ P(x) &= -5x^2 + 20x + 300 \end{aligned}$$

By setting $P(x) = 0$ and factoring, we can verify that the zeros of the function occur at -6 and 10.

$$\begin{aligned} 0 &= -5x^2 + 20x + 300 \\ 0 &= -5(x^2 - 4x - 60) \\ 0 &= -5(x + 6)(x - 10) \end{aligned}$$

Therefore, $x = -6$ or $x = 10$.

Examples

Example 2

d. Determine the equation of the profit function. Use this equation to verify the graph of the profit function that was produced in part c).

Solution

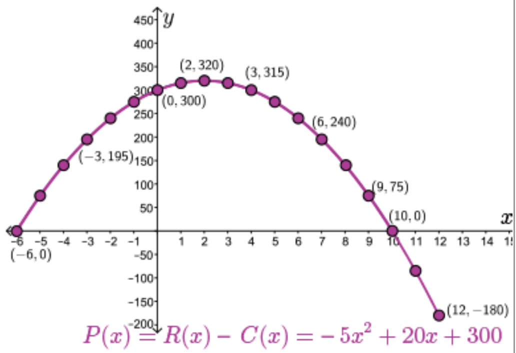
$P(x) = -5x^2 + 20x + 300$

By completing the square, we can show the vertex is a maximum at $(2, 320)$.

$$\begin{aligned} P(x) &= -5(x^2 - 4x) + 300 \\ &= -5(x^2 - 4x + 4) - 4(-5) + 300 \\ &= -5(x - 2)^2 + 320 \end{aligned}$$

The graph of $P(x)$ can be obtained by

- reflecting the graph of $y = x^2$ in the x-axis,
- applying a vertical stretch about the x-axis by a factor of 5, and
- translating the graph right 2 units and up 320 units.



Summary

Two functions can be combined by addition or subtraction to create a new, often more complex, function:

- The sum of two functions f and g is a function defined by $(f + g)(x) = f(x) + g(x)$.
- The difference of two functions f and g is a function defined by $(f - g)(x) = f(x) - g(x)$.
- The domain of the sum or difference function is the set of all real numbers in the domain of both f and g.
- The graph of $f + g$ can be obtained from the graphs of f and g by adding corresponding y-coordinates for values of x in the domain of both f and g.
- The graph of $f - g$ can be obtained from the graphs of f and g by subtracting the y-coordinates of g from the corresponding y-coordinates of f for values of x in the domain of both f and g.
- The behaviour of the graph of each (non-zero) function f and g will, in some way, influence the behaviour of the graph of $f + g$ and $f - g$.
- The graph of $f - g$ is the reflection of the graph of $g - f$ in the x-axis.
- For non-zero functions, we have the following:
  - The sum or difference of two even functions is an even function.
  - The sum or difference of two odd functions is an odd function.
  - The sum or difference of an even and odd function is a function that is neither even nor odd.