Solving Polynomial Inequalities

Solving Linear Inequalities

Review

Solve $5x - 3 \geq 7x - 9,\ x \in \mathbb{R}$.

Solution

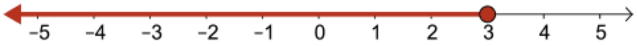
The steps involved in solving a linear inequality are similar to the steps used to solve a linear equation, with one exception: when multiplying or dividing both sides of the inequality by a negative value, the inequality condition must be reversed.

$$\begin{aligned} 5x - 3 &\geq 7x - 9 \\ 5x - 7x &\geq -9 + 3 \\ -2x &\geq -6 \\ \frac{-2x}{-2} &\leq \frac{-6}{-2} \\ x &\leq 3 \end{aligned}$$

Therefore, the solution is $\{x \mid x \leq 3,\ x \in \mathbb{R}\}$.

The solution can also be stated using interval notation: $x \in (-\infty, 3],\ x \in \mathbb{R}$.

It can also be represented graphically using a real number line as shown:



Solving Polynomial Inequalities

Example 1

Solve $x^2 - 3x > 10,\ x \in \mathbb{R}$.

Algebraic Solution

We begin as we do with a quadratic equation: bring all terms of the inequality to one side, leaving zero on the other side. Then, we factor the quadratic.

$$\begin{aligned} x^2 - 3x - 10 &> 0 \\ (x - 5)(x + 2) &> 0 \end{aligned}$$

Now, a product of two factors is positive in two cases.

Case 1. Both factors are positive.

$$\begin{aligned} x - 5 > 0 &\text{ and } x + 2 > 0 \\ x > 5 &\text{ and } x > -2 \\ x &> 5 \end{aligned}$$

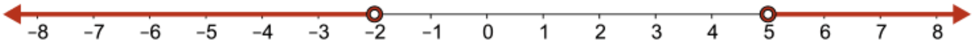
OR

Case 2. Both factors are negative.

$$\begin{aligned} x - 5 < 0 &\text{ and } x + 2 < 0 \\ x < 5 &\text{ and } x < -2 \\ x &< -2 \end{aligned}$$

Therefore, the solution to this quadratic inequality is $\{x \mid x < -2 \text{ or } x > 5,\ x \in \mathbb{R}\}$.

In interval notation, this is $x \in (-\infty, -2) \cup (5, \infty)$. This can also be illustrated using a number line as shown:



Solving Polynomial Inequalities

Example 1

Solve $x^2 - 3x > 10,\ x \in \mathbb{R}$.

Graphical Solution

$$x^2 - 3x > 10$$
$$x^2 - 3x - 10 > 0$$
$$(x - 5)(x + 2) > 0$$

Let $f(x) = (x - 5)(x + 2)$, graph the function, and determine when $f(x) > 0$.

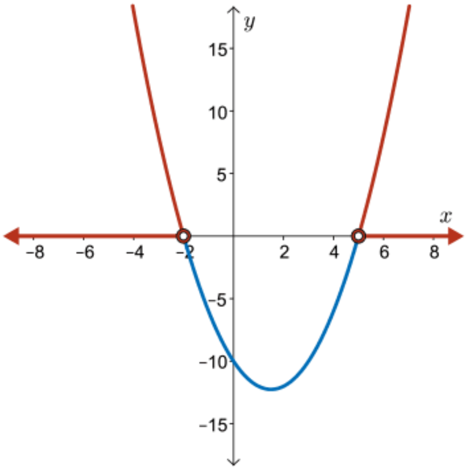
The x-intercepts are $x = -2$ and $x = 5$.

The leading coefficient is positive, so the parabola opens upward.

From the graph of $f(x) = (x - 5)(x + 2)$, we see that $f(x) > 0$ (the graph is above the x-axis) when

$$x < -2 \text{ or } x > 5$$

This confirms the solution found using an algebraic "case" approach.



Solving Polynomial Inequalities

Example 2

Solve $2x^2 - x^3 \geq 2 - x,\ x \in \mathbb{R}$.

Algebraic Solution

We begin by arranging the terms on one side of the inequality.

$$2x^2 - x^3 \geq 2 - x$$
$$-x^3 + 2x^2 + x - 2 \geq 0$$
$$x^3 - 2x^2 - x + 2 \leq 0$$

We factor by grouping:

$$x^2(x - 2) - 1(x - 2) \leq 0$$
$$(x - 2)(x^2 - 1) \leq 0$$
$$(x - 2)(x - 1)(x + 1) \leq 0$$

Solving Polynomial Inequalities

Example 2

Solve $2x^2 - x^3 \geq 2 - x,\ x \in \mathbb{R}$.

Algebraic Solution

$$(x-2)(x-1)(x+1) \leq 0$$

- There are 3 factors: $(x-2)$, $(x-1)$, and $(x+1)$. We need to detemine when the product of these three factors is less than or equal to zero.
- The product of these three factors will be zero when one of the factors is zero and the product will be negative when an odd number of the factors are negative.

Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7	Case 8
$x-2<0$	$x-2<0$	$x-2<0$	$x-2<0$	$x-2>0$	$x-2>0$	$x-2>0$	$x-2>0$
$x-1<0$	$x-1<0$	$x-1>0$	$x-1>0$	$x-1<0$	$x-1<0$	$x-1>0$	$x-1>0$
$x+1<0$	$x+1>0$	$x+1<0$	$x+1>0$	$x+1<0$	$x+1>0$	$x+1<0$	$x+1>0$

Solving Polynomial Inequalities

Example 2

Solve $2x^2 - x^3 \geq 2 - x,\ x \in \mathbb{R}$.

Algebraic Solution

$$(x-2)(x-1)(x+1) \leq 0$$

- There are 3 factors: $(x-2)$, $(x-1)$, and $(x+1)$. We need to detemine when the product of these three factors is less than or equal to zero.
- The product of these three factors will be zero when one of the factors is zero and the product will be negative when an odd number of the factors are negative.

- The sign of the value of each linear factor changes at the “zero” value of the factor.

  For example, $x-2$ is positive when $x > 2$ and negative when $x < 2$.
- We begin by identifying the zero values for the factors: $x = \pm 1$ and $x = 2$.

  We will use these values to identify intervals where changes in the sign of the factors may occur.

Solving Polynomial Inequalities

Example 2

Solve $2x^2 - x^3 \geq 2 - x,\ x \in \mathbb{R}$.

Algebraic Solution

$$(x-2)(x-1)(x+1) \leq 0$$

We set up a table as shown:

	$x=-1$	$x=1$	$x=2$	
	$x<-1$	$-1<x<1$	$1<x<2$	$x>2$
$x-2$	$-$	$-$	$-$	$+$
$x-1$	$-$	$-$	$+$	$+$
$x+1$	$-$	$+$	$+$	$+$
$(x-2)(x-1)(x+1)$	$-$	$+$	$-$	$+$

- We choose a test value from within each interval, substitute this value into each factor, and identify if the factor is positive or negative in that interval.
  For the interval $x < -1$, we use $x = -2$ as a test value.
  For the interval $-1 < x < 1$, we use $x = 0$ as a test value.
  For the interval $1 < x < 2$, we use $x = 1.5$ as a test value.
  For the interval $x > 2$, we use $x = 3$ as a test value.
- We determine the sign of $(x-2)(x-1)(x+1)$ by determining the sign of the product of the factors.

Solving Polynomial Inequalities

Example 2

Solve $2x^2 - x^3 \geq 2 - x,\ x \in \mathbb{R}$.

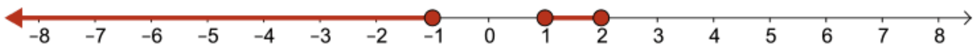
Algebraic Solution

$$(x-2)(x-1)(x+1) \leq 0$$

We set up a table as shown:

	$x=-1$	$x=1$	$x=2$	
	$x<-1$	$-1<x<1$	$1<x<2$	$x>2$
$x-2$	$-$	$-$	$-$	$+$
$x-1$	$-$	$-$	$+$	$+$
$x+1$	$-$	$+$	$+$	$+$
$(x-2)(x-1)(x+1)$	$-$	$+$	$-$	$+$

Therefore, the solution is $\{x \mid x \leq -1 \text{ or } 1 \leq x \leq 2,\ x \in \mathbb{R}\}$.



$$x \in (-\infty, -1] \cup [1, 2],\ x \in \mathbb{R}$$

Solving Polynomial Inequalities

Example 2

Solve $2x^2 - x^3 \geq 2 - x,\ x \in \mathbb{R}$.

Graphical Solution

$$\begin{aligned} 2x^2 - x^3 &\geq 2 - x \\ -x^3 + 2x^2 + x - 2 &\geq 0 \\ x^3 - 2x^2 - x + 2 &\leq 0 \\ (x-2)(x-1)(x+1) &\leq 0 \end{aligned}$$

Let $f(x) = (x-2)(x-1)(x+1)$.

We will sketch the graph and determine when $f(x) \leq 0$.

- We identify the zeros: $x = \pm 1$ and $x = 2$. All three zeros are of multiplicity one, so the graph will pass through each zero.
- Since the function is cubic with a positive leading coefficient, it will have "opposite" end behaviours with

$$y \to -\infty \text{ as } x \to -\infty$$

and

$$y \to \infty \text{ as } x \to \infty$$

Solving Polynomial Inequalities

Example 2

Solve $2x^2 - x^3 \geq 2 - x,\ x \in \mathbb{R}$.

Graphical Solution

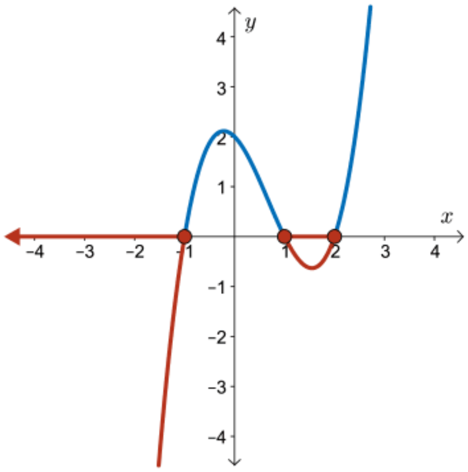
We begin our sketch in the third quadrant, passing through each of the zeros and ending in the first quadrant.

From the graph of $f(x) = (x-2)(x-1)(x+1)$, we see that $f(x) \leq 0$ when

$$x \leq -1 \text{ or } 1 \leq x \leq 2$$

Therefore, the solution is

$$\{x \mid x \leq -1 \text{ or } 1 \leq x \leq 2,\ x \in \mathbb{R}\}$$



Solving Polynomial Inequalities

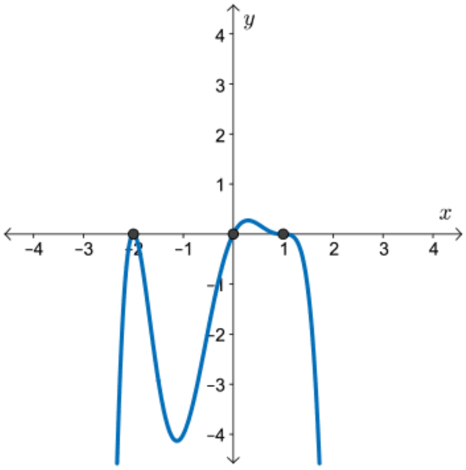
Example 3

Solve $-0.5x(x+2)^2(x-1)^3 \geq 0$.

Graphical Solution

Let $y = -0.5x(x+2)^2(x-1)^3$.

- We have a 6^{th} degree polynomial function with a negative leading coefficient. The graph will have “same” end behaviours with
$$y \to -\infty \text{ as } x \to \pm\infty$$
- There are 3 zeros:
  - A zero at $x = -2$ of multiplicity two, resulting in a turning point at this zero.
  - A zero at $x = 0$ of multiplicity one, so the curve will pass directly through the x-axis at the origin.
  - A zero of multiplicity three at $x = 1$, so the graph will pass through the x-axis with a point of inflection.



Solving Polynomial Inequalities

Example 3

Solve $-0.5x(x+2)^2(x-1)^3 \geq 0$.

Graphical Solution

$$y = -0.5x(x+2)^2(x-1)^3$$

Using the graph, we see that $y \geq 0$ (that is, the curve is on or above the x-axis) when

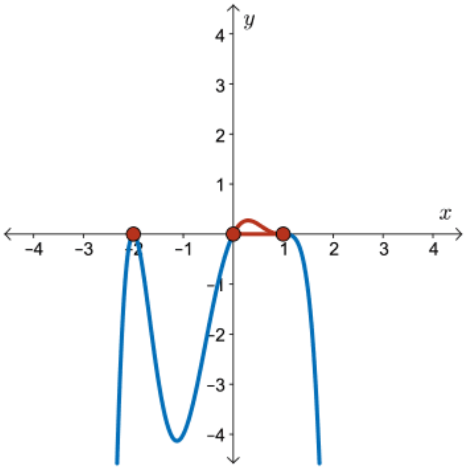
$$x = -2 \text{ or } 0 \leq x \leq 1$$

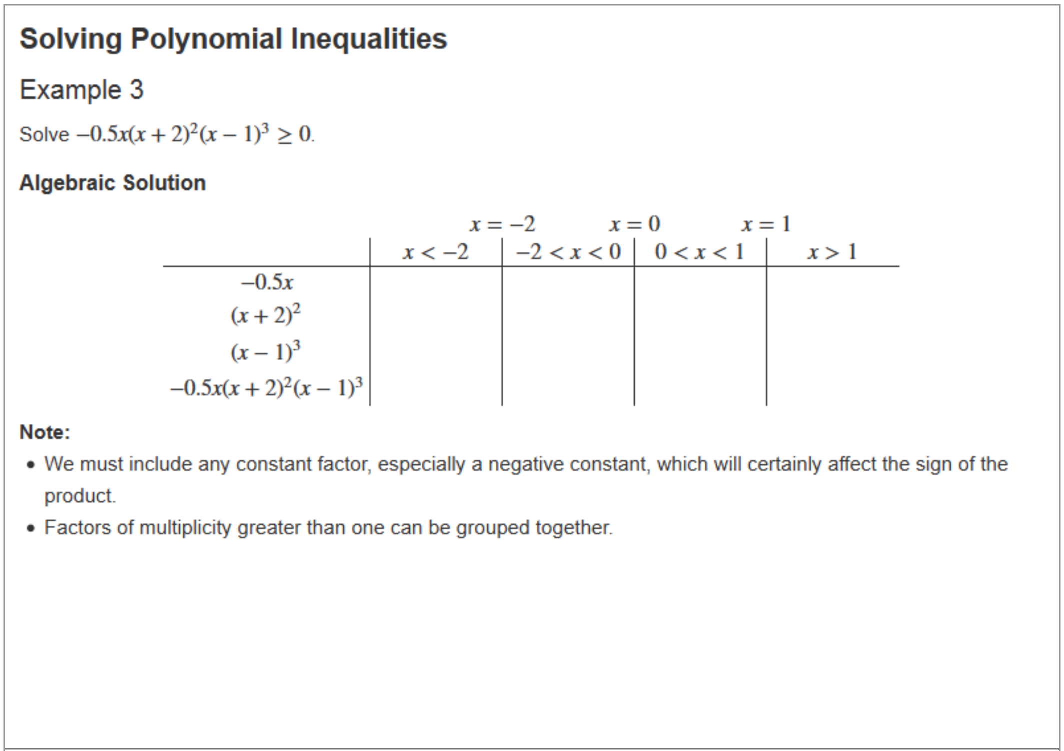
Therefore,

$$-0.5x(x+2)^2(x-1)^3 \geq 0$$

when

$$x \in \{-2\} \cup [0, 1],\ x \in \mathbb{R}$$





Solving Polynomial Inequalities

Example 3

Solve $-0.5x(x+2)^2(x-1)^3 \geq 0$.

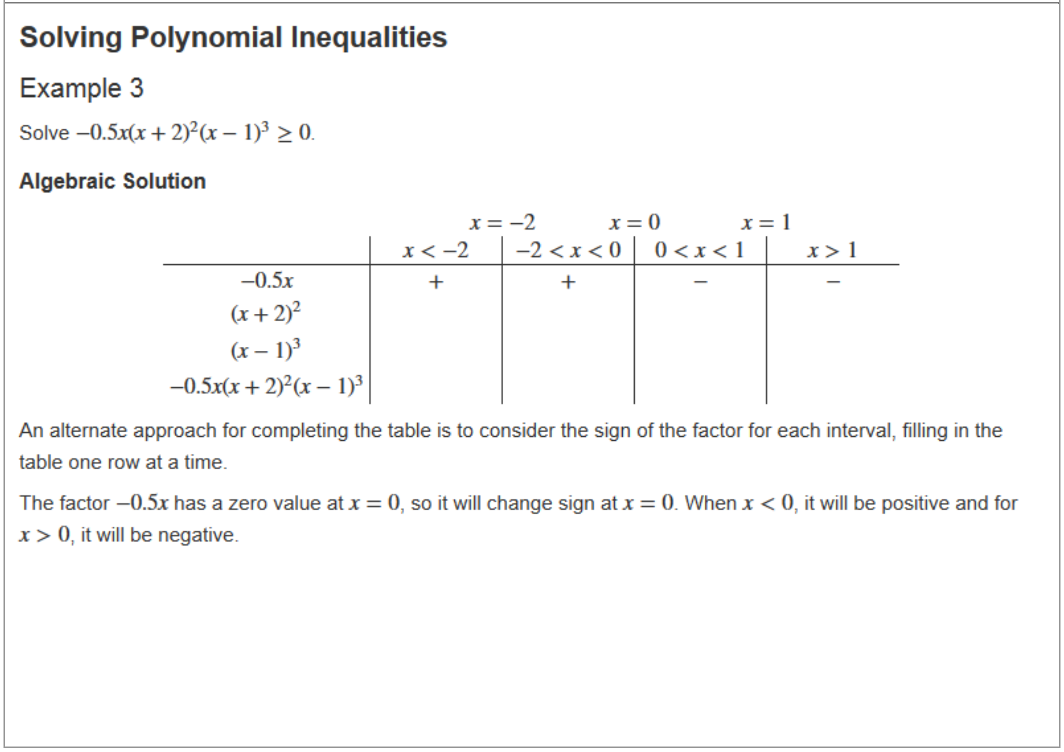
Algebraic Solution

$x = -2$ $x = 0$ $x = 1$

	$x < -2$	$-2 < x < 0$	$0 < x < 1$	$x > 1$
$-0.5x$				
$(x+2)^2$				
$(x-1)^3$				
$-0.5x(x+2)^2(x-1)^3$				

Note:

- We must include any constant factor, especially a negative constant, which will certainly affect the sign of the product.
- Factors of multiplicity greater than one can be grouped together.



Solving Polynomial Inequalities

Example 3

Solve $-0.5x(x+2)^2(x-1)^3 \geq 0$.

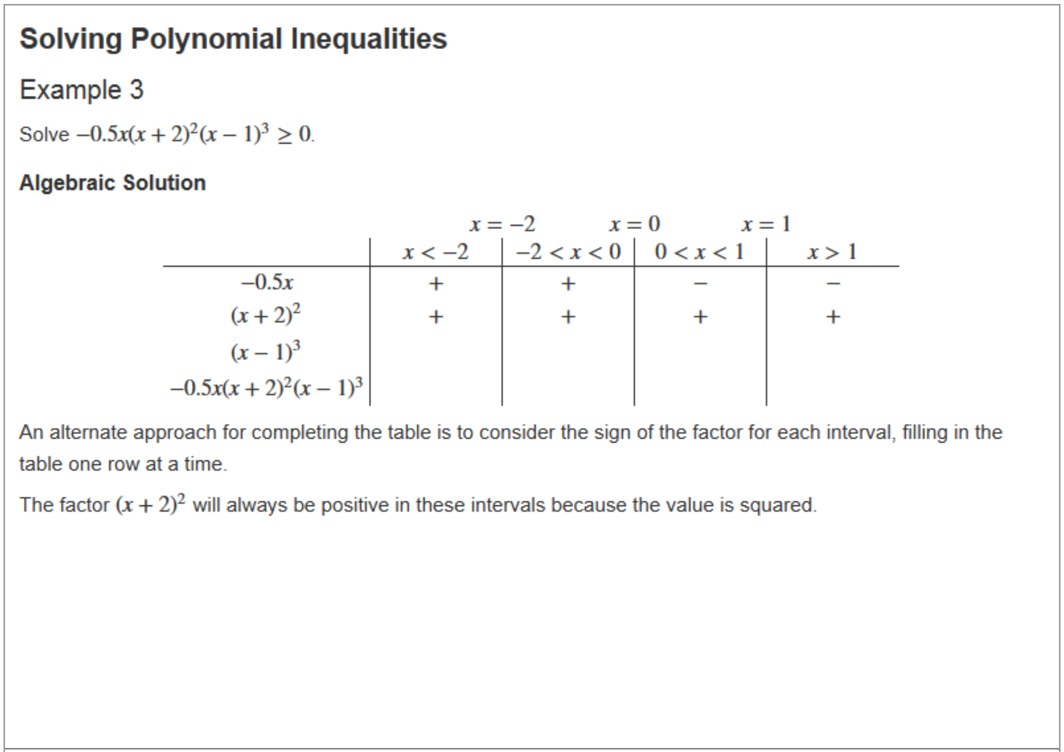
Algebraic Solution

$x = -2$ $x = 0$ $x = 1$

	$x < -2$	$-2 < x < 0$	$0 < x < 1$	$x > 1$
$-0.5x$	$+$	$+$	$-$	$-$
$(x+2)^2$				
$(x-1)^3$				
$-0.5x(x+2)^2(x-1)^3$				

An alternate approach for completing the table is to consider the sign of the factor for each interval, filling in the table one row at a time.

The factor $-0.5x$ has a zero value at $x = 0$, so it will change sign at $x = 0$. When $x < 0$, it will be positive and for $x > 0$, it will be negative.



Solving Polynomial Inequalities

Example 3

Solve $-0.5x(x+2)^2(x-1)^3 \geq 0$.

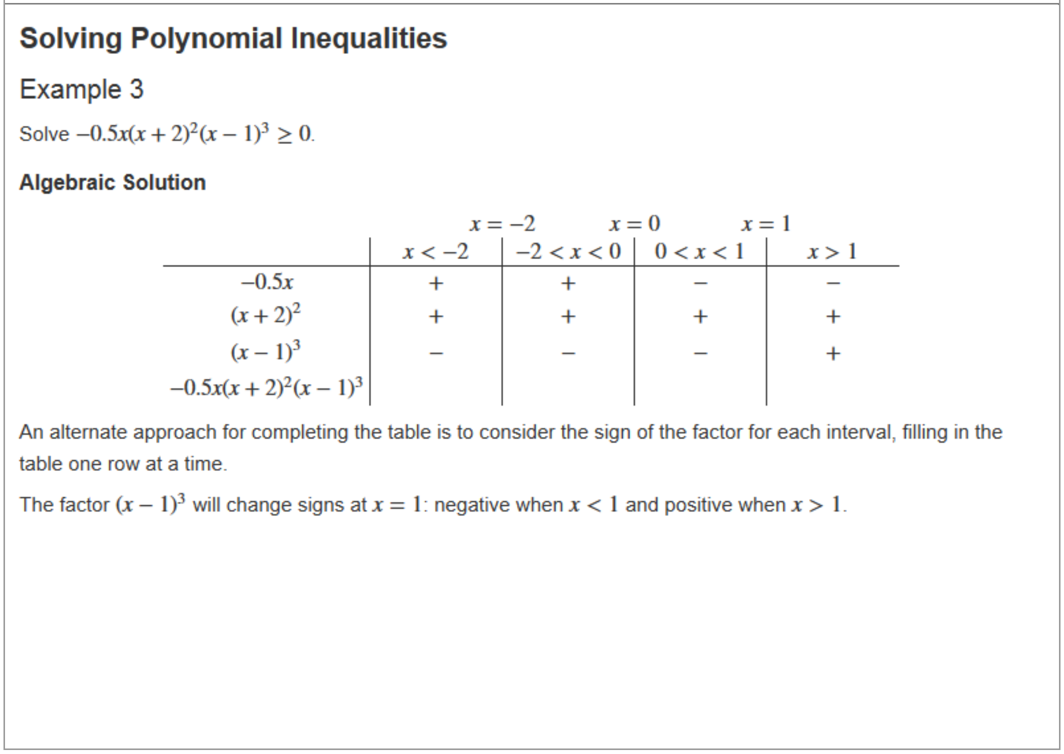
Algebraic Solution

$x = -2$ $x = 0$ $x = 1$

	$x < -2$	$-2 < x < 0$	$0 < x < 1$	$x > 1$
$-0.5x$	$+$	$+$	$-$	$-$
$(x+2)^2$	$+$	$+$	$+$	$+$
$(x-1)^3$				
$-0.5x(x+2)^2(x-1)^3$				

An alternate approach for completing the table is to consider the sign of the factor for each interval, filling in the table one row at a time.

The factor $(x+2)^2$ will always be positive in these intervals because the value is squared.



Solving Polynomial Inequalities

Example 3

Solve $-0.5x(x+2)^2(x-1)^3 \geq 0$.

Algebraic Solution

$x = -2$ $x = 0$ $x = 1$

	$x < -2$	$-2 < x < 0$	$0 < x < 1$	$x > 1$
$-0.5x$	$+$	$+$	$-$	$-$
$(x+2)^2$	$+$	$+$	$+$	$+$
$(x-1)^3$	$-$	$-$	$-$	$+$
$-0.5x(x+2)^2(x-1)^3$				

An alternate approach for completing the table is to consider the sign of the factor for each interval, filling in the table one row at a time.

The factor $(x-1)^3$ will change signs at $x = 1$: negative when $x < 1$ and positive when $x > 1$.

Solving Polynomial Inequalities

Example 3

Solve $-0.5x(x+2)^2(x-1)^3 \geq 0$.

Algebraic Solution

$x = -2$ $x = 0$ $x = 1$

	$x < -2$	$-2 < x < 0$	$0 < x < 1$	$x > 1$
$-0.5x$	$+$	$+$	$-$	$-$
$(x+2)^2$	$+$	$+$	$+$	$+$
$(x-1)^3$	$-$	$-$	$-$	$+$
$-0.5x(x+2)^2(x-1)^3$	$-$	$-$	$+$	$-$

An alternate approach for completing the table is to consider the sign of the factor for each interval, filling in the table one row at a time.

We now determine the sign of the $-0.5x(x+2)^2(x-1)^3$ by determining the product of the factors in each interval.

Therefore, the solution to $-0.5x(x+2)^2(x-1)^3 \geq 0$ is $\{x \mid x = -2 \text{ or } 0 \leq x \leq 1,\ x \in \mathbb{R}\}$

Solving Polynomial Inequalities

Example 4

A cubic function, $y = f(x)$, has a turning point at $(-2, 0)$, an x-intercept at $x = 1$, and $f(-1) = 4$. Determine all values of x such that $0 < f(x) < 8$.

Solution

To determine the equation, we can use the zeros and an additional point on the curve.

- There is a zero at $x = -2$ of multiplicity two. Therefore, $(x+2)^2$ is a factor of the function.
- There is a zero at $x = 1$ of multiplicity one. Therefore, $(x-1)$ is a factor of the function.

Let $f(x) = a(x-1)(x+2)^2$.
Since $f(-1) = 4$,

$$\begin{aligned} 4 &= a(-1-1)(-1+2)^2 \\ 4 &= a(-2)(1)^2 \\ a &= -2 \end{aligned}$$

Therefore,

$$f(x) = -2(x-1)(x+2)^2$$

To determine all values of x such that $0 < f(x) < 8$, solve $0 < -2(x-1)(x+2)^2 < 8$. This translates into solving the inequalities,

$$0 < -2(x-1)(x+2)^2 \text{ and } -2(x-1)(x+2)^2 < 8$$

individually.

Solving Polynomial Inequalities

Example 4

A cubic function, $y = f(x)$, has a turning point at $(-2, 0)$, an x-intercept at $x = 1$, and $f(-1) = 4$. Determine all values of x such that $0 < f(x) < 8$.

Solution

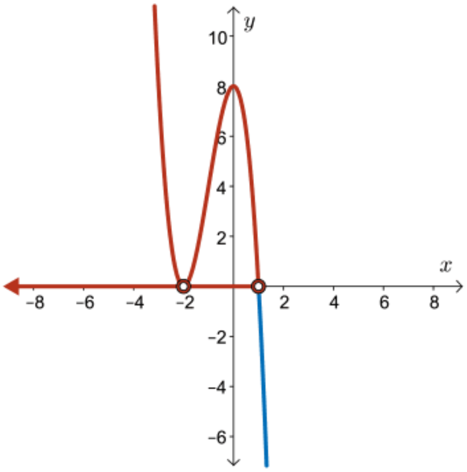
We solve $-2(x-1)(x+2)^2 > 0$ graphically.
Using the zeros and end behaviour, we can sketch the graph of $f(x) = -2(x-1)(x+2)^2$.
$f(x) > 0$ when

$$x < -2$$

or

$$-2 < x < 1$$

Therefore, $\{x \mid x < 1, x \neq -2,\ x \in \mathbb{R}\}$.



Solving Polynomial Inequalities

Example 4

A cubic function, $y = f(x)$, has a turning point at $(-2, 0)$, an x-intercept at $x = 1$, and $f(-1) = 4$. Determine all values of x such that $0 < f(x) < 8$.

Solution

Now, solve $-2(x-1)(x+2)^2 < 8$ algebraically.

$$\begin{aligned}(-2x+2)(x^2+4x+4) &< 8\\ -2x^3 - 8x^2 - 8x + 2x^2 + 8x + 8 &< 8\\ -2x^3 - 6x^2 + 8 &< 8\\ -2x^3 - 6x^2 &< 0\\ -2x^2(x+3) &< 0\\ x^2(x+3) &> 0\end{aligned}$$

We can solve this inequality using an interval table.

	$x < -3$	$-3 < x < 0$	$x > 0$
	$x = -3$	$x = 0$	
x^2	+	+	+
$x+3$	−	+	+
$x^2(x+3)$	−	+	+

Therefore, $x^2(x+3) > 0$ when $-3 < x < 0$ or $x > 0$.

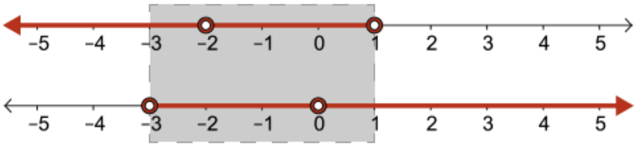
Solving Polynomial Inequalities

Example 4

A cubic function, $y = f(x)$, has a turning point at $(-2, 0)$, an x-intercept at $x = 1$, and $f(-1) = 4$. Determine all values of x such that $0 < f(x) < 8$.

Solution

The solution to $0 < f(x) < 8$ is all $x \in \mathbb{R}$ such that $x < 1, x \neq -2$ and $x > -3, x \neq 0$.
To determine which values of x satisfy both sets of conditions, it may be helpful to look at the solution of each inequality on a number line, as shown:



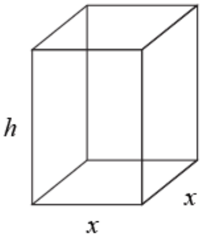
Since both sets of conditions must be satisfied, the grey area highlighting the overlap in the two sets of conditions helps us identify the solution.
Therefore, the solution is $\{x \mid -3 < x < 1,\ x \neq -2,\ x \neq 0,\ x \in \mathbb{R}\}$.

Solving Polynomial Inequalities

Example 5

Wire is bent to form the frame of a square-based rectangular prism. If 36 cm of wire is used, determine the restrictions on the length of the square base if the volume of the prism must be less than 20 cm^3.

Solution



Let x represent the length of the side of the square base $(x > 0)$ and h represent the height of the rectangular prism $(h > 0)$.
If 36 cm of wire is used to form the frame of the prism, then

$$\begin{aligned} 8x + 4h &= 36 \\ h &= 9 - 2x \end{aligned}$$

Now,

$$\begin{aligned} V(x) &= x^2 h \\ V(x) &= x^2(9 - 2x) \end{aligned}$$

To determine the restrictions on x, we must solve for $0 < V(x) < 20$.
Specifically, the inequality is $0 < x^2(9 - 2x) < 20$, which gives two inequalities to solve:

$$0 < x^2(9 - 2x) \text{ and } x^2(9 - 2x) < 20$$

Solving Polynomial Inequalities

Example 5

Wire is bent to form the frame of a square-based rectangular prism. If 36 cm of wire is used, determine the restrictions on the length of the square base if the volume of the prism must be less than 20 cm^3.

Solution

To solve for $x^2(9-2x) > 0$, we determine when $9-2x > 0$ since $x^2 > 0$ for all $x \neq 0$.

$$\begin{aligned} 9-2x &> 0 \\ -2x &> -9 \\ x &< 4.5 \end{aligned}$$

Therefore, $x < 4.5$, but $x > 0$, so $0 < x < 4.5$.

Solve $x^2(9-2x) < 20$:

$$\begin{aligned} -2x^3+9x^2-20 &< 0 \\ 2x^3-9x^2+20 &> 0 \end{aligned}$$

Solving Polynomial Inequalities

Example 5

Wire is bent to form the frame of a square-based rectangular prism. If 36 cm of wire is used, determine the restrictions on the length of the square base if the volume of the prism must be less than 20 cm^3.

Solution

Let $g(x) = 2x^3 - 9x^2 + 20$.
Note that $g(2) = 0$. Therefore, $(x-2)$ is a factor of $g(x)$.
Using synthetic division,

$$\begin{array}{r|rrrr} 2 & 2 & -9 & 0 & 20 \\ & & 4 & -10 & -20 \\ \hline & 2 & -5 & -10 & 0 \end{array}$$

$$g(x) = (x-2)(2x^2-5x-10)$$

Thus, it remains to find the zeros of $g(x) = (x-2)(2x^2-5x-10)$.
We quickly identify $x = 2$.

$$\begin{aligned} x &= \frac{5 \pm \sqrt{(-5)^2 - 4(2)(-10)}}{2(2)} \\ x &= \frac{5 \pm \sqrt{105}}{4} \\ x &\approx -1.31, 3.81 \end{aligned}$$

Solving Polynomial Inequalities

Example 5

Wire is bent to form the frame of a square-based rectangular prism. If 36 cm of wire is used, determine the restrictions on the length of the square base if the volume of the prism must be less than 20 cm^3.

Solution

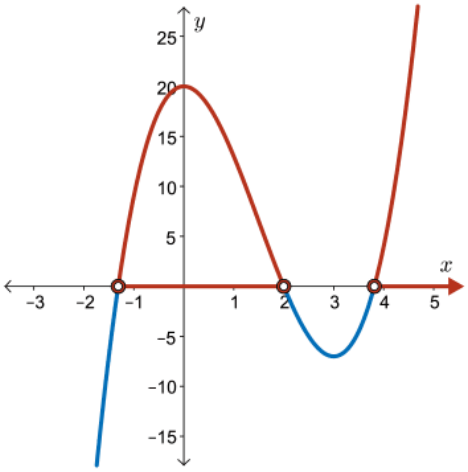
Using these zeros and the end behaviour of the cubic, $g(x) = 2x^3 - 9x^2 + 20$, we can sketch the graph and determine when $g(x) > 0$; that is, when the function is above the x-axis.

$g(x) > 0$ when

$$\frac{5-\sqrt{105}}{4} < x < 2$$

or

$$x > \frac{5+\sqrt{105}}{4}$$



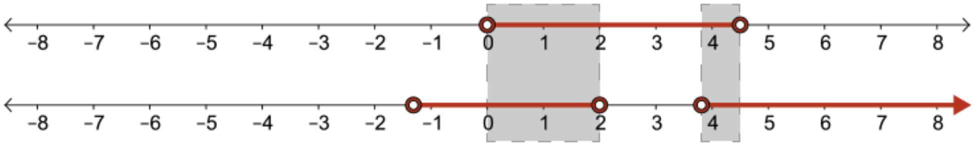
Solving Polynomial Inequalities

Example 5

Wire is bent to form the frame of a square-based rectangular prism. If 36 cm of wire is used, determine the restrictions on the length of the square base if the volume of the prism must be less than 20 cm^3.

Solution

So, $0 < V(x) < 20$ when x satisfies $(0 < x < 4.5)$ and $\left(\frac{5-\sqrt{105}}{4} < x < 2 \text{ or } x > \frac{5+\sqrt{105}}{4}\right)$.



Therefore,

$$0 < x < 2 \text{ or } \frac{5+\sqrt{105}}{4} < x < 4.5$$

Solving Polynomial Inequalities

Example 5

Wire is bent to form the frame of a square-based rectangular prism. If 36 cm of wire is used, determine the restrictions on the length of the square base if the volume of the prism must be less than 20 cm^3.

Solution

Another approach to determine when

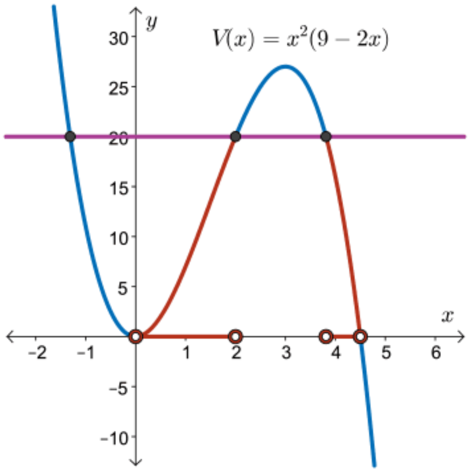
$$0 < V(x) < 20$$

would be to graph $y = V(x)$ and $y = 20$ and identify the values of x when the graph of $y = V(x)$ is between $y = 20$ and the x-axis.

Note: $V(x) = 20$ at $x = 2$ and $x = \dfrac{5 \pm \sqrt{105}}{4}$, and x is the side of the base, so $x > 0$.

Hence, $0 < x < 2$ or $\dfrac{5 + \sqrt{105}}{4} < x < 4.5$.

Therefore, the volume is between 0 and 20 cm^3 when the length of the base is between 0 cm and 2 cm, or between $\frac{5+\sqrt{105}}{4}\text{ cm}$ and 4.5 cm.



Summary

- When multiplying or dividing both sides of the inequality by a negative value, the inequality condition must be reversed.
- When solving a factorable polynomial inequality of degree 2 or greater, arrange the terms to one side of the inequality condition, with 0 on the other side. Factor the polynomial expression to identify when the expression is equal to 0.
- An interval table can be created using the zeros and factors of the polynomial to help identify the intervals when the polynomial expression is positive or negative in value, thus determining the solution to the inequality.
- An alternate approach is to graph the corresponding polynomial function, using the factored expression, and identify the intervals when the graph is above (positive) or below (negative) the x-axis.
- Number lines can also be used to provide a visual aid when solving inequalities.