The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

Characteristics of Polynomial Functions

In This Module

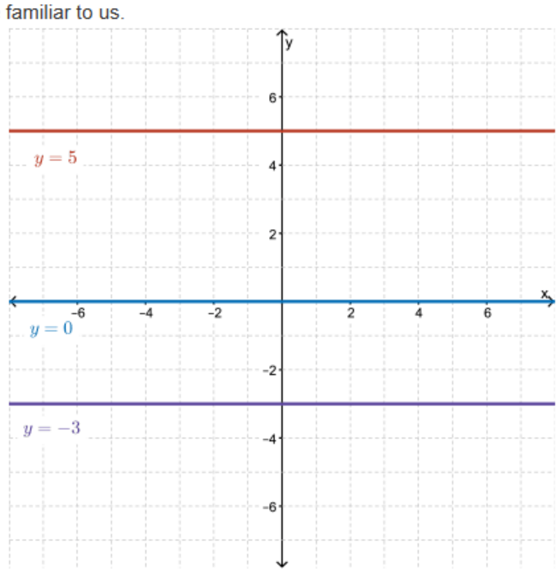
- We will review and explore the behaviour of the graphs of various polynomial functions ranging from degree 0 to degree 6.
- The focus will be on determining the possible number of turning points, the possible number of x-intercepts, the end behaviour of the function's graph, and how the end behaviour is influenced by the leading coefficient of the function (the coefficient of the highest degree term).

Constant Functions

Let's first discuss some polynomial functions that are familiar to us.

Polynomial functions of degree 0 are constant functions of the form $y = a, a \in \mathbb{R}$.
Their graphs are horizontal lines with a y-intercept at $(0, a)$.
The end behaviours of these graphs can be summarized with the statement, "as $x \to \pm\infty, y = a$."

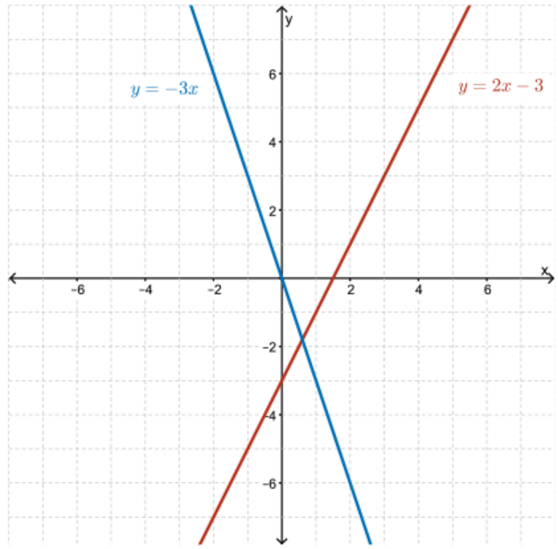
Constant functions have no turning points and no zeros, except for the case of $y = 0$, which is the equation of the x-axis and therefore has an infinite number of zeros.



Linear Functions

Polynomial functions of degree 1 are linear functions of the form $y = ax + b, a \neq 0$.
We know that a provides the slope of the line and b is the y-intercept.
We also know that lines do not have turning points.

If we consider the graphs of the lines $y = 2x - 3$ and $y = -3x$, we see that a line will have one x-intercept if the line is not horizontal (as seen previously).

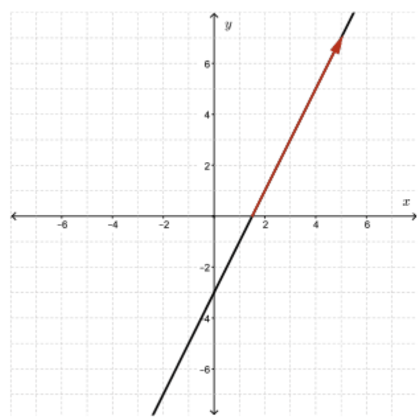


Linear Functions

The end behaviours of a linear function are dependent on the slope of the line, which is given by the leading coefficient.

When the slope is positive, the end behaviour can be described in the following way:

- y approaches a large negative value as x approaches a large negative value.
- y will approach a large positive value as x approaches a large positive value.

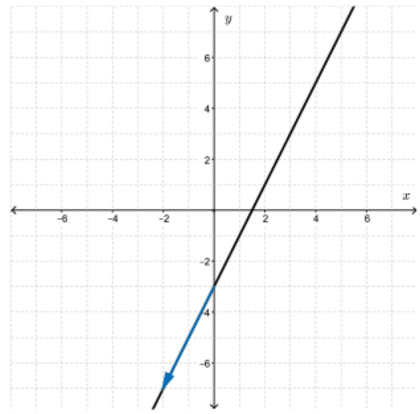


Linear Functions

The end behaviours of a linear function are dependent on the slope of the line, which is given by the leading coefficient.

When $a > 0$,

$y \to -\infty$ as $x \to -\infty$
$y \to \infty$ as $x \to \infty$



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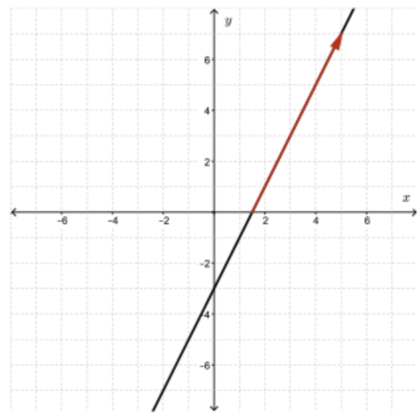

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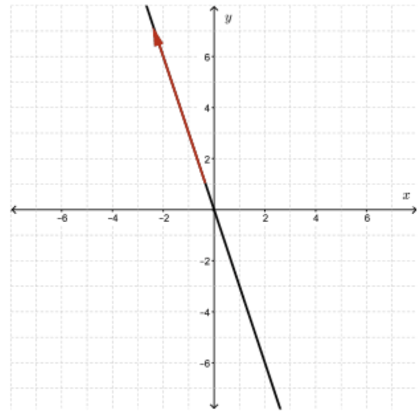
When $a < 0$,

$$y \to \infty \text{ as } x \to -\infty$$
$$y \to -\infty \text{ as } x \to \infty$$



Linear Functions

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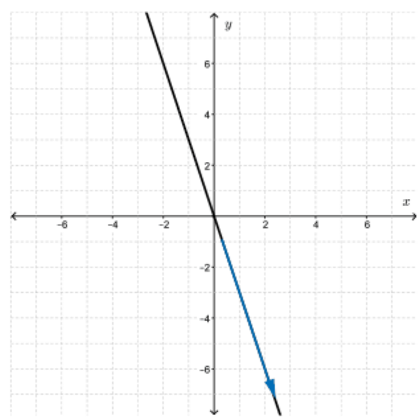
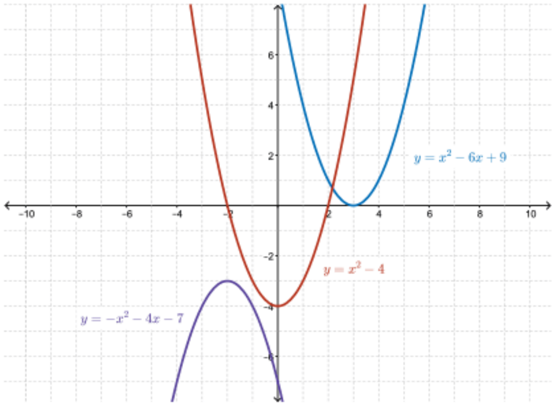


Quadratic Functions

The quadratic function, $y = ax^2 + bx + c$, is a polynomial function of degree 2.

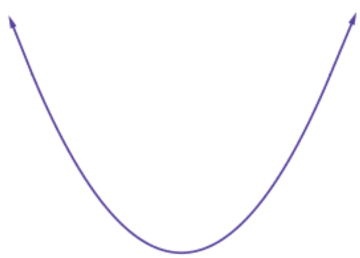
The graph of a quadratic function (a parabola) has one turning point, which is an absolute maximum or minimum point on the curve.



We can see from the graphs of the quadratic functions shown that a parabola may have no x-intercept, 1 x-intercept, or 2 x-intercepts depending on where the vertex is and the direction in which the parabola opens.

Quadratic Functions

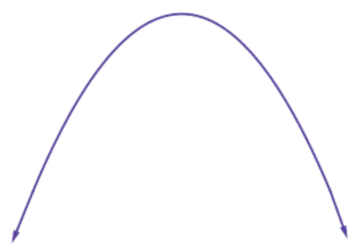
If the leading coefficient is positive ($a > 0$), then the parabola will open upward.



When $a > 0$,

$$y \to \infty \text{ as } x \to -\infty$$
$$y \to \infty \text{ as } x \to \infty$$

If the leading coefficient is negative ($a < 0$), then the parabola will open downward.



When $a < 0$,

$$y \to -\infty \text{ as } x \to -\infty$$
$$y \to -\infty \text{ as } x \to \infty$$

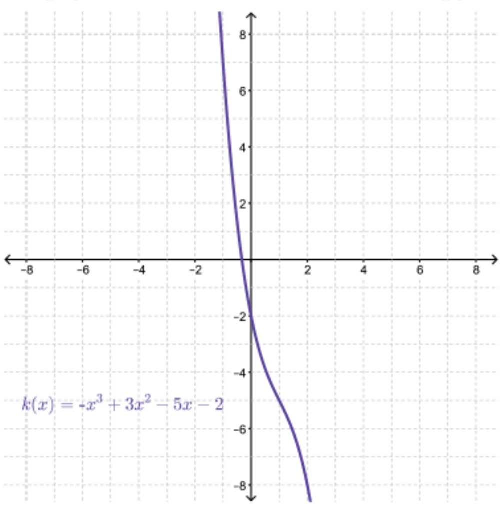
The range of a quadratic function is dependent on the maximum or minimum value of the function and the direction of opening.

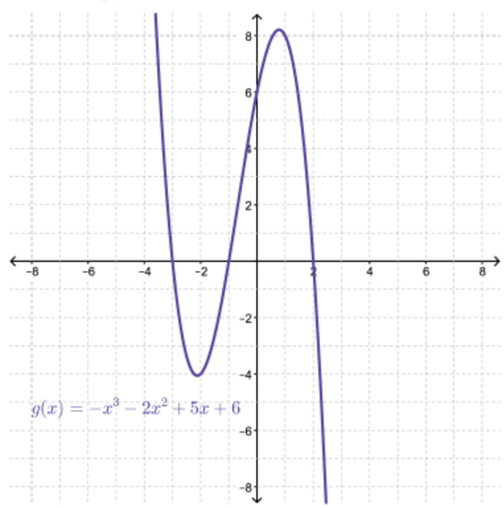
Maple Investigation

We will now investigate the graphs of higher degree polynomial functions.
We will focus on key features including: the number of turning points, the number of x-intercepts, and the end behaviours of the function.

Cubic Functions: $y = ax^3 + bx^2 + cx + d, a \neq 0$

The graph of a cubic function has either no turning point or two turning points.

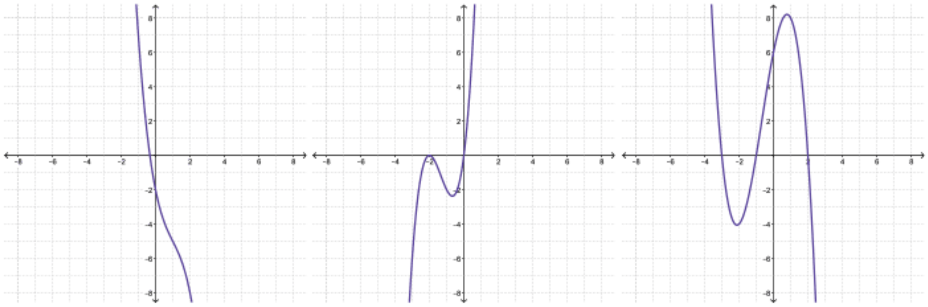




If the graph has no turning point, it will have a point of inflection similar to that of $y = x^3$.

Cubic Functions: $y = ax^3 + bx^2 + cx + d, a \neq 0$

It is possible for the graph of a cubic function to have 1, 2, or 3 x-intercepts depending on whether the graph has a turning point and the position of the graph on the axis.

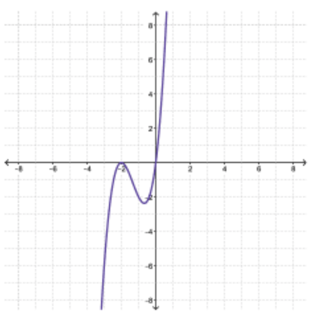


Cubic Functions: $y = ax^3 + bx^2 + cx + d, a \neq 0$

The graph of a cubic function has opposite end behaviours.

If the leading coefficient is positive, then

$$y \to -\infty \text{ as } x \to -\infty$$
$$y \to \infty \text{ as } x \to \infty$$

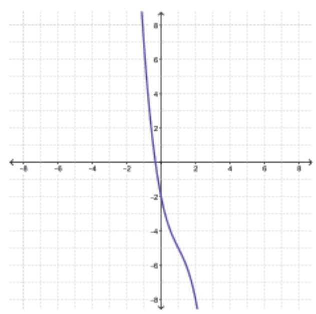


$$y = 2x^3 + 8x^2 + 8x$$

The graph will begin in the third quadrant and end in the first quadrant.

If the leading coefficient is negative, then

$$y \to \infty \text{ as } x \to -\infty$$
$$y \to -\infty \text{ as } x \to \infty$$

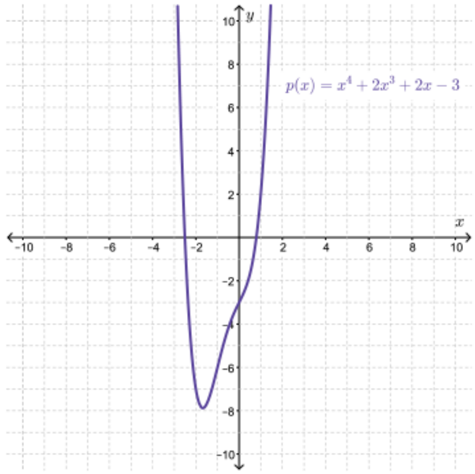


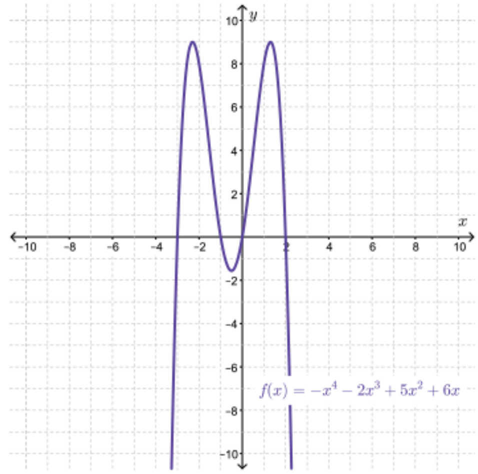
$$y = -x^3 + 3x^2 - 5x - 2$$

The graph will start in the second quadrant and end in the fourth quadrant.

Quartic Functions: $y = ax^4 + bx^3 + cx^2 + dx + e, a \neq 0$

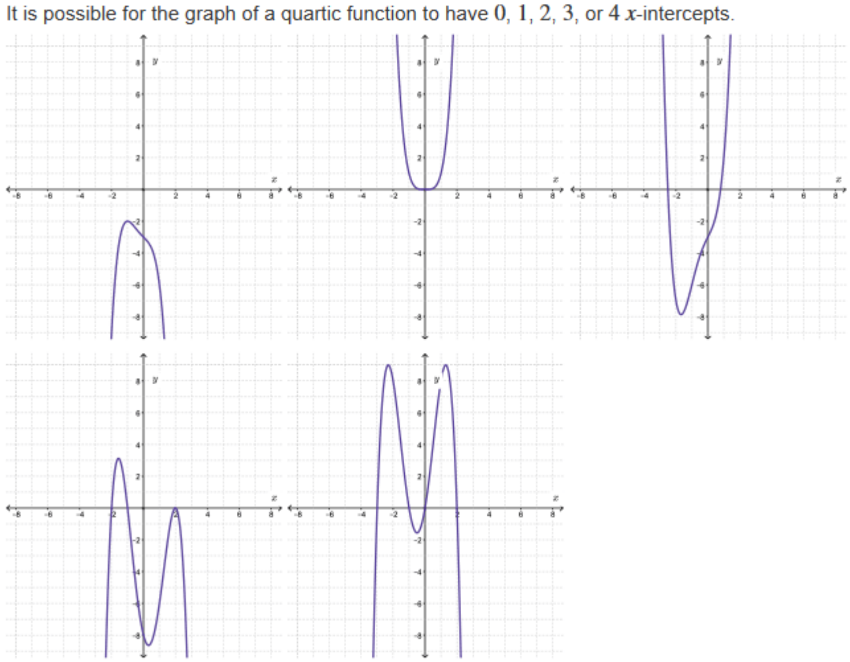
The graph of a quartic function has either one turning point or three turning points.





Quartic Functions: $y = ax^4 + bx^3 + cx^2 + dx + e, a \neq 0$

It is possible for the graph of a quartic function to have 0, 1, 2, 3, or 4 x-intercepts.

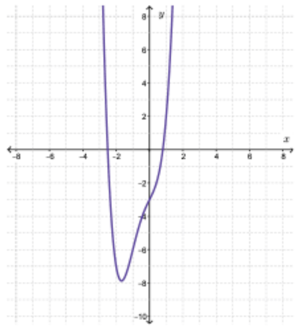


Quartic Functions: $y = ax^4 + bx^3 + cx^2 + dx + e, a \neq 0$

The graph of a quartic function has "same" end behaviours, similar to that of a quadratic function.

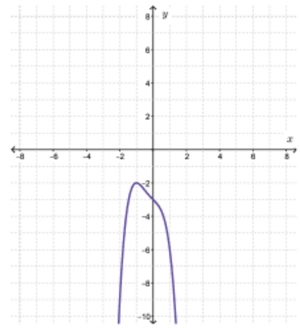
If the leading coefficient is positive, then

$y \to \infty$ as $x \to \pm\infty$



If the leading coefficient is negative, then

$y \to -\infty$ as $x \to \pm\infty$



This end behaviour is similar to that of a parabola.

Maple Investigation

Does this pattern continue for 5^{th} and 6^{th} degree polynomial functions?

Observations

The following are characteristics of the graphs of n^{th} degree polynomial functions where n is odd:

- The graph will have end behaviours similar to that of a linear function.
  If the leading coefficient is positive, then $y \to -\infty$ as $x \to -\infty$ and $y \to \infty$ as $x \to \infty$.
  If the leading coefficient is negative, then $y \to \infty$ as $x \to -\infty$ and $y \to -\infty$ as $x \to \infty$.
- The graph will have an even number of turning points to a maximum of $n - 1$ turning points.
  For example, a 5^{th} degree polynomial function may have 0, 2, or 4 turning points.
- The graph will have at least one x-intercept to a maximum of n x-intercepts.
  If we consider a 5^{th} degree polynomial function, it must have at least 1 x-intercept and a maximum of 5 x-intercepts.

Observations

The following are characteristics of the graphs of n^{th} degree polynomial functions where n is even:

- The graph will have end behaviours similar to that of a parabola, often described as same end behaviours.
  If the leading coefficient is positive, then $y \to \infty$ as $x \to \pm\infty$.
  If the leading coefficient is negative, then $y \to -\infty$ as $x \to \pm\infty$.
- The graph will have an odd number of turning points to a maximum of $n - 1$ turning points.
  A 6^{th} degree polynomial function will have a possible 1, 3, or 5 turning points.
- The graph will have an absolute maximum or minimum point due to the nature of the end behaviour.
  The range of these functions will depend on the absolute maximum or minimum value and the direction of the end behaviours.
- The graph will have a minimum of 0 to a maximum of n x-intercepts.
  For example, a 6^{th} degree polynomial function will have a minimum of 0 x-intercepts and a maximum of 6 x-intercepts.

Examples

Example 1

a. Describe the end behaviour of $y = -2x^3 + 5x + 4$.

Solution

The function is a 3^{rd} degree polynomial function.
The degree of the polynomial is odd and the leading coefficient is negative.
Therefore, the end behaviours are opposite and described by $y \to \infty$ as $x \to -\infty$ and $y \to -\infty$ as $x \to \infty$.

The end behaviour of a polynomial function is determined by the highest degree term, $-2x^3$.
As x becomes large in value, the other terms of the polynomial, $5x$ and 4, become insignificant in value compared to the value of $-2x^3$.
When x is a large negative number, $-2x^3$ is a large positive value.
When x is a large positive number, $-2x^3$ is a large negative value.

Examples

Example 1

b. Describe the end behaviour of $y = 5x^4 + 3x^3 - 2x^2 + 3x + 1$.

Solution

The function is a 4^{th} degree polynomial function.
The degree of the polynomial is even and the leading coefficient is positive.
Therefore, the end behaviours are in the same direction and described by $y \to \infty$ as $x \to \pm\infty$.
The end behaviour is similar to that of a parabola with a positive leading coefficient.

The end behaviour of this polynomial function is determined by the term $5x^4$.
Again, as x becomes large in value, the other terms of the polynomial, $3x^3$, $-2x^2$, $3x$, and 1, become insignificant in value compared to the value of $5x^4$.
When x is a large negative number, $5x^4$ is a large positive value.
When x is a large positive number, $5x^4$ is, again, a large positive value.

Examples

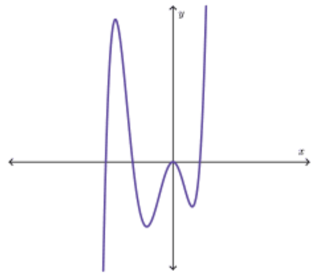
Example 2

Given the shape of a graph of the polynomial function, determine the least possible degree of the function and state the sign of the leading coefficient.

This function has opposite end behaviours, so it is an odd degree polynomial function.

Its end behaviour is similar to that of a line with a positive slope, so the leading coefficient is positive.

It has 4 turning points, so it must be at least a 5^{th} degree polynomial.



Examples

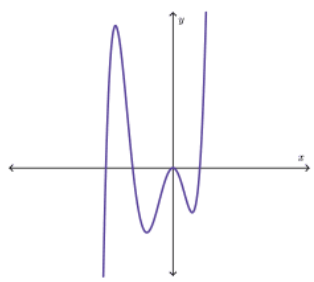
Example 2

Given the shape of a graph of the polynomial function, determine the least possible degree of the function and state the sign of the leading coefficient.

Note: It is possible for a higher odd degree polynomial function to have a similar shape.

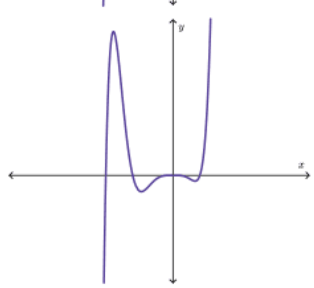
The actual function is a 5^{th} degree polynomial.

$$y = x^2(x-2)(x+3)(x+5)$$



Here is a graph of a 7^{th} degree polynomial with a similar shape.

$$y = x^4(x-2)(x+3)(x+5)$$



Examples

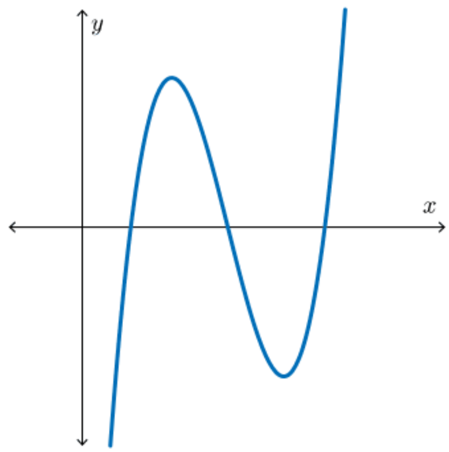
Example 3

Sketch a possible graph of a 3^{rd} degree polynomial function with a positive leading coefficient that has:

a. three x-intercepts **b.** two x-intercepts **c.** one x-intercept

Solution

a. A cubic function can have zero or two turning points. To create a function with three x-intercepts, we will need to work with two turning points.
A positive leading coefficient means that $y \to -\infty$ as $x \to -\infty$ and $y \to \infty$ as $x \to \infty$, similar to a line with a positive slope.
Start with the shape and then place the x-axis such that the curve crosses the x-axis three times.



Examples

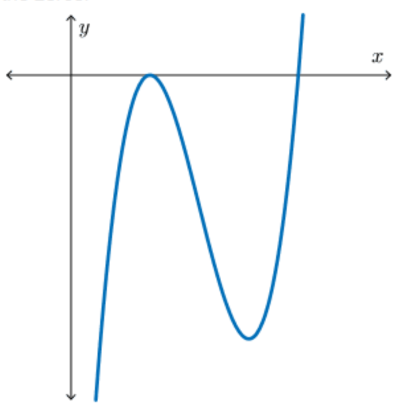
Example 3

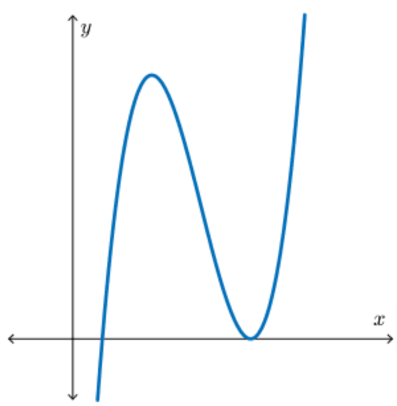
Sketch a possible graph of a 3^{rd} degree polynomial function with a positive leading coefficient that has:

a. three x-intercepts **b.** two x-intercepts **c.** one x-intercept

Solution

b. To create the cubic with two x-intercepts, we would need to move the axis such that a turning point is at one of the zeros.





The y-axis may be placed anywhere.

Examples

Example 3

Sketch a possible graph of a 3^{rd} degree polynomial function with a positive leading coefficient that has:

a. three x-intercepts **b.** two x-intercepts **c.** one x-intercept

Solution

c. To create the cubic with one x-intercept, we can simply move the x-axis again, or work with a cubic that has no turning point.