Even and Odd Polynomial Functions

In This Module

- We will investigate the symmetry of higher degree polynomial functions.
- We will generalize a rule that will assist us in recognizing even and odd symmetry, when it occurs in a polynomial function.

Symmetry in Polynomials

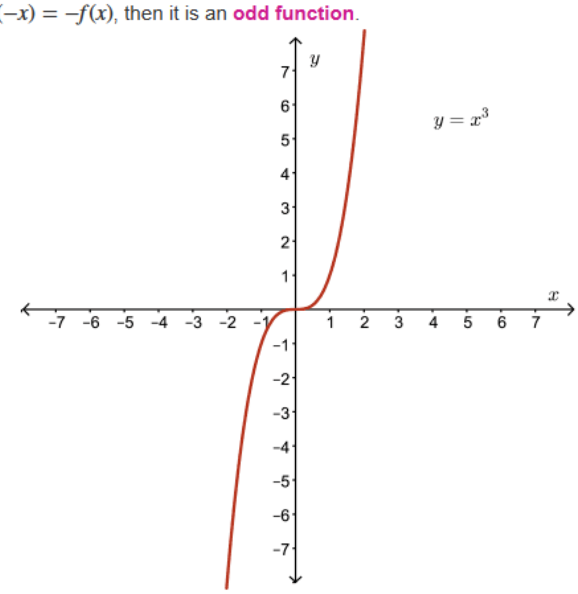
Recall, a function can be even, odd, or neither depending on its symmetry.

If a function is symmetric about the y-axis, then the function is an **even function** and $f(-x) = f(x)$.

If a function is symmetric about the origin, that is $f(-x) = -f(x)$, then it is an **odd function**.

The cubic function, $y = x^3$, an odd degree polynomial function, is an odd function.

That is, the function is symmetric about the origin.



Symmetry in Polynomials

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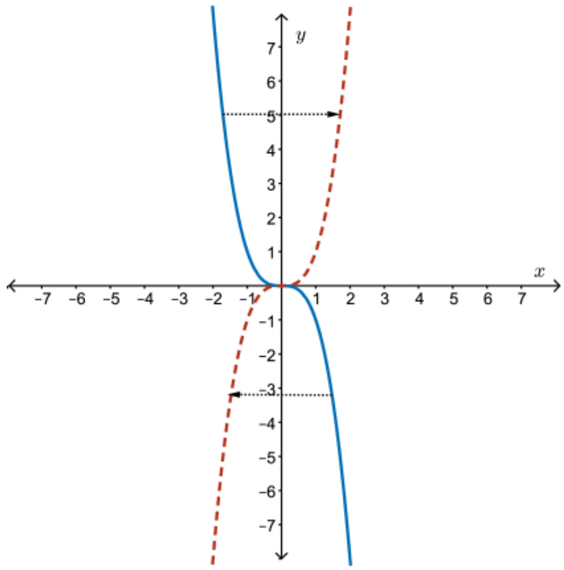
If the graph of the function is reflected in the x-axis, followed by a reflection in the y-axis, it will map onto itself.



Symmetry in Polynomials

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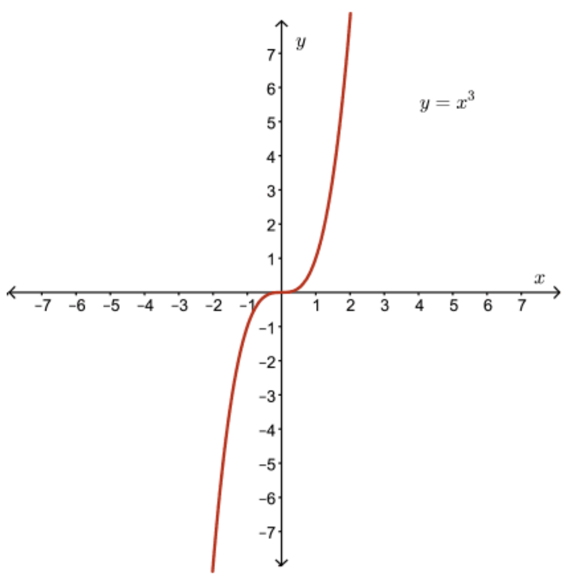
Algebraically,

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

Since $f(-x) = -f(x)$, $y = x^3$ is an odd function.

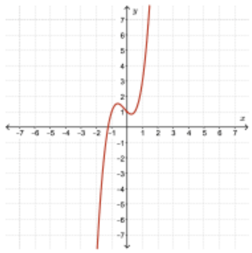
Is this the case for all cubic functions?



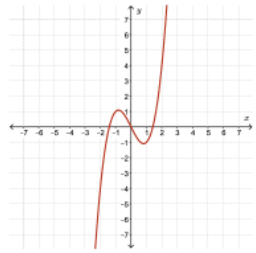
Symmetry in Polynomials

Consider the following cubic functions and their graphs.

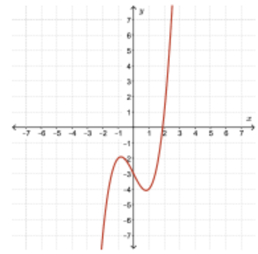
(1) $y = 2x^3 + x^2 - x + 1$



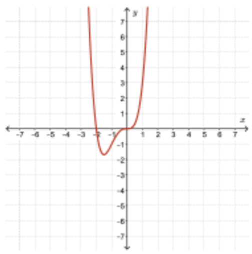
(2) $y = x^3 - 2x$



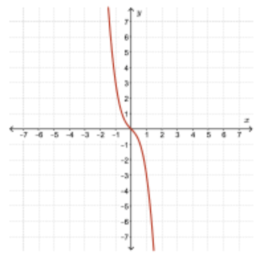
(3) $y = x^3 - 2x - 3$



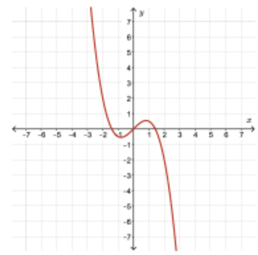
(4) $y = x^3 - 2x^2$



(5) $y = -2x^3 - x$



(6) $y = -\frac{1}{2}x^3 + x$



Symmetry in Polynomials

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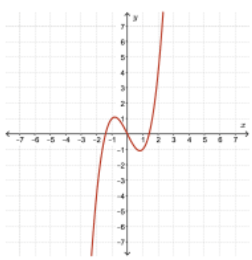

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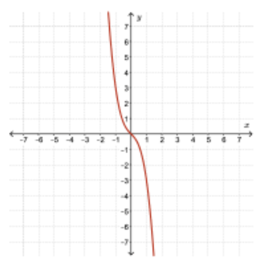

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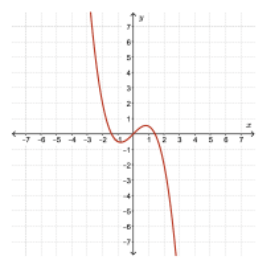


What do these functions have in common?

They have a term of degree 3 and a term of degree 1; that is, an x^3 term and an x term.

The other 3 functions defined by

$$y = 2x^3 + x^2 - x + 1 \qquad y = x^3 - 2x - 3 \qquad y = x^3 - 2x^2$$

are neither even nor odd.

Along with an odd degree term, x^3, these functions also have terms of even degree; that is, an x^2 term and/or a constant term of degree 0.

It appears an odd polynomial must have only odd degree terms.

Symmetry in Polynomials

If we consider the general 3^{rd} degree polynomial function,

$$f(x) = ax^3 + bx^2 + cx + d$$

then

$$\begin{aligned} f(-x) &= a(-x)^3 + b(-x)^2 + c(-x) + d \\ &= -ax^3 + bx^2 - cx + d \\ &\neq f(x) \end{aligned}$$

for any values of a, b, c, or d since $a \neq 0$.
Therefore, a cubic function is never an even function.

Now, $-f(x) = -ax^3 - bx^2 - cx - d$, so $f(-x) = -f(x)$ when $b = 0$ and $d = 0$, that is when $f(x) = ax^3 + cx$.
Therefore, cubic functions of the form $f(x) = ax^3 + cx$, $a \neq 0$, are odd functions.

Symmetry in Polynomials

Similarly, it should follow that even polynomial functions would have only even degree terms.
If we consider the general 4^{th} degree polynomial function,

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e$$

then

$$\begin{aligned} f(-x) &= a(-x)^4 + b(-x)^3 + c(-x)^2 + d(-x) + e \\ &= ax^4 - bx^3 + cx^2 - dx + e \end{aligned}$$

Setting the coefficients $b = 0$ and $d = 0$ will result in $f(-x) = f(x)$, and hence an even function.
Therefore, quartic functions of the form $f(x) = ax^4 + cx^2 + e$, $a \neq 0$, are even functions.
And,

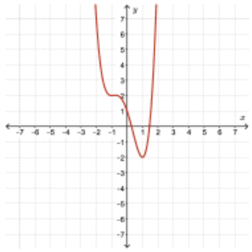
$$\begin{aligned} -f(x) &= -ax^4 - bx^3 - cx^2 - dx - e \\ &\neq f(-x) \end{aligned}$$

for any values of a, b, c, or d since $a \neq 0$.
Therefore, a quartic function is never an odd function.

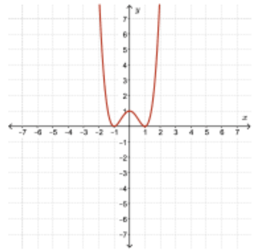
Symmetry in Polynomials

To illustrate the results graphically, we compare the following functions:

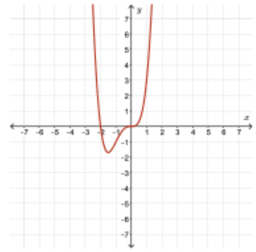
(1) $y = x^4 + x^3 - 2x^2 - 3x + 1$



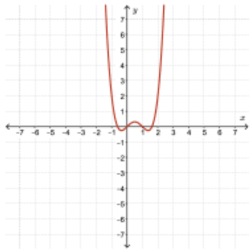
(2) $y = x^4 - 2x^2 + 1$



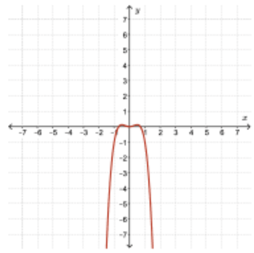
(3) $y = x^4 + 2x^3$



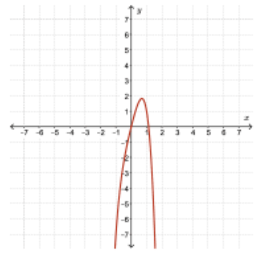
(4) $y = x^4 - 2x^3 + x$



(5) $y = -2x^4 + x^2$



(6) $y = -2x^4 - x^2 + 4x$



In Summary

- A polynomial function is an **even function** if and only if each of the terms of the function is of an even degree.
- A polynomial function is an **odd function** if and only if each of the terms of the function is of an odd degree.
- The graphs of even degree polynomial functions will never have odd symmetry.
- The graphs of odd degree polynomial functions will never have even symmetry.

Note: The polynomial function $f(x) = 0$ is the one exception to the above set of rules. This function is both an even function (symmetrical about the y-axis) and an odd function (symmetrical about the origin).

Examples

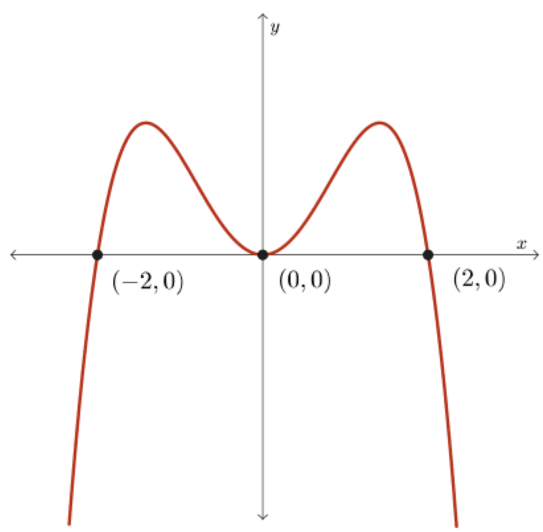
Example 1

Sketch the graph of the function $y = -2x^4 + 8x^2$.
What do we know about this function?

- The function is an even degree polynomial with a negative leading coefficient. Therefore, $y \to -\infty$ as $x \to \pm\infty$.
- Since all of the terms of the function are of an even degree, the function is an even function. Therefore, the function is symmetrical about the y-axis.
- The function in factored form is $y = -2x^2(x^2 - 4) = -2x^2(x - 2)(x + 2)$. Therefore, the zeros of the function are at $x = \pm 2$ and $x = 0$ (multiplicity 2).

Examples

Sketch



Examples

Example 2

a. Show that every polynomial function can be expressed as the sum of an even and an odd polynomial function.

Solution

Let $P(x)$ be any polynomial function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

where the coefficients $a_0, a_1, a_2, a_3, \ldots, a_n$ are real numbers, $n \geq 0$, and $n \in Z$.

If n is even, then

$$\begin{aligned} P(x) = \ & \left(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_0\right) && \text{(even degree terms)} \\ & + \left(a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_1 x\right) && \text{(odd degree terms)} \end{aligned}$$

showing $P(x)$ as the sum of an even function and an odd function.

If n is odd, then

$$\begin{aligned} P(x) = \ & \left(a_n x^n + a_{n-2}x^{n-2} + \cdots + a_3 x^3 + a_1 x\right) && \text{(odd degree terms)} \\ & + \left(a_{n-1}x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_2 x^2 + a_0\right) && \text{(even degree terms)} \end{aligned}$$

showing $P(x)$ as the sum of an odd function and an even function.

Note: If $P(x)$ is an even function (or odd function), then $P(x)$ can be expressed as the sum of $P(x)$ and $f(x) = 0$. (Recall: $f(x) = 0$ is both an even and odd function.)

Examples

Example 2

b. Prove that every function can be expressed as the sum of an even and odd function.

Solution

Let $f(x)$ be any function. Observe that

$$\begin{aligned} f(x) &= \frac{2f(x)}{2} \\ &= \frac{f(x) + f(x)}{2} \\ &= \frac{f(x) + 0 + f(x)}{2} \\ &= \frac{f(x) + (f(-x) - f(-x)) + f(x)}{2} \\ &= \frac{(f(x) + f(-x)) + (f(x) - f(-x))}{2} \\ &= \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \end{aligned}$$

Let $g(x) = \dfrac{f(x) + f(-x)}{2}$ and $h(x) = \dfrac{f(x) - f(-x)}{2}$; therefore, $f(x) = g(x) + h(x)$.

Examples

Example 2

b. Prove that every function can be expressed as the sum of an even and odd function.

Solution

Let $g(x) = \dfrac{f(x)+f(-x)}{2}$ and $h(x) = \dfrac{f(x)-f(-x)}{2}$; therefore, $f(x) = g(x) + h(x)$.

Now, $g(-x) = \dfrac{f(-x)+f(-(-x))}{2} = \dfrac{f(-x)+f(x)}{2} = g(x)$, so $g(x)$ is even.

And

$$\begin{aligned} h(-x) &= \frac{f(-x)-f(-(-x))}{2} \\ &= \frac{f(-x)-f(x)}{2} \\ &= \frac{-\left(f(x)-f(-x)\right)}{2} \\ &= -\frac{f(x)-f(-x)}{2} \\ &= -h(x) \end{aligned}$$

so $h(x)$ is odd.

Therefore, any function, $f(x)$, can be expressed as the sum of an even and odd function.