Finite Differences Of Polynomial Functions

In This Module

- We will investigate the behaviour of finite differences for higher degree polynomial functions.

Investigating Finite Differences of Polynomial Functions

A line has a constant rate of change, in other words, a constant slope.
Consider the table of values for the linear function $y = 3x - 2$.

x	$y = 3x - 2$	1^{st} Difference Δy
-2	-8	$-5 - (-8) = 3$
-1	-5	$-2 - (-5) = 3$
0	-2	$1 - (-2) = 3$
1	1	$4 - 1 = 3$
2	4	

The x values in this table are in increments of 1, that is $\Delta x = 1$.
To calculate the first differences, denoted by Δy, we will compute the changes or differences in the y values of the function.
The first differences are equal, with a constant value of 3. Therefore, $\Delta y = 3$ and $\Delta x = 1$.
So, the slope of the line $\frac{\Delta y}{\Delta x}$ is 3.

Finite Differences of Quadratic Functions

For a quadratic function, the rate of change of y as x changes is variable.
The parabola does not have a constant slope.

x	$y = -x^2 + 3x + 1$	1st Difference Δy	2nd Difference $\Delta^2 y$
-2	-9	$-3 - (-9) = 6$	$4 - 6 = -2$
-1	-3	$1 - (-3) = 4$	$2 - 4 = -2$
0	1	$3 - 1 = 2$	$0 - 2 = -2$
1	3	$3 - 3 = 0$	
2	3		

With quadratic functions, the first differences, Δy, are variable.
But the difference in the first differences, that is, the second differences, denoted by $\Delta^2 y$, are constant.

Finite Differences of Quadratic Functions

Here are further examples to support this fact.

x	$y = x^2 + 4x - 3$	Δy	$\Delta^2 y$
-2	-7	1	2
-1	-6	3	2
0	-3	5	2
1	2	7	
2	9		

x	$y = 5x^2 - 4x$	Δy	$\Delta^2 y$
-2	28	-19	10
-1	9	-9	10
0	0	1	10
1	1	11	
2	12		

x	$y = -3x^2 + 10$	Δy	$\Delta^2 y$
-2	-2	9	-6
-1	7	3	-6
0	10	-3	-6
1	7	-9	
2	-2		

What might you expect to find with the finite differences of cubic or quartic functions?

Finite Differences of Cubic Functions

Consider the following finite difference tables for four cubic functions.

x	$y = x^3$	Δy	$\Delta^2 y$	$\Delta^3 y$
−2	−8	7	−6	6
−1	−1	1	0	6
0	0	1	6	
1	1	7		
2	8			

x	$y = -3x^3 + 2x^2$	Δy	$\Delta^2 y$	$\Delta^3 y$
−2	32	−27	22	−18
−1	5	−5	4	−18
0	0	−1	−14	
1	−1	−15		
2	−16			

x	$y = -4x^3 + 2x + 1$	Δy	$\Delta^2 y$	$\Delta^3 y$
−2	29	−26	24	−24
−1	3	−2	0	−24
0	1	−2	−24	
1	−1	−26		
2	−27			

x	$y = 2x^3 + x$	Δy	$\Delta^2 y$	$\Delta^3 y$
−2	−18	15	−12	12
−1	−3	3	0	12
0	0	3	12	
1	3	15		
2	18			

The third differences, $\Delta^3 y$, are constant for these 3^{rd} degree functions.

Finite Differences of Quartic Functions

Consider the following finite difference tables for these quartic functions.

x	$y = x^4$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
−2	16	−15	14	−12	24
−1	1	−1	2	12	24
0	0	1	14	36	
1	1	15	50		
2	16	65			
3	81				

x	$y = 2x^4 - x^2 - 1$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
−2	27	−27	26	−24	48
−1	0	−1	2	24	48
0	−1	1	26	72	
1	0	27	98		
2	27	125			
3	152				

x	$y = -3x^4 + x - 1$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
−2	−51	46	−42	36	−72
−1	−5	4	−6	−36	−72
0	−1	−2	−42	−108	
1	−3	−44	−150		
2	−47	−194			
3	−241				

x	$y = 3x^4 + x^2 - 1$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
−2	51	−48	44	−36	72
−1	3	−4	8	36	72
0	−1	4	44	108	
1	3	48	152		
2	51	200			
3	251				

Once again, 4^{th} degree polynomials have constant fourth differences, denoted by $\Delta^4 y$.

Observations

Is there more to this pattern?

The last two quartic examples suggest a connection between the leading coefficient, a, of a polynomial and the value of the constant difference.

x	$y=-3x^4+x-1$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
−2	−51	46	−42	36	−72
−1	−5	4	−6	−36	−72
0	−1	−2	−42	−108	
1	−3	−44	−150		
2	−47	−194			
3	−241				

x	$y=3x^4+x^2-1$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
−2	51	−48	44	−36	72
−1	3	−4	8	36	72
0	−1	4	44	108	
1	3	48	152		
2	51	200			
3	251				

If you consider the last two quartic functions, the leading coefficients are opposite in sign (−3 and 3) and the value of the constant difference for each are opposite in sign (−72 and 72).

Observations

Let's compile the data for the preceding examples and compare the value of the leading coefficient to that of the constant difference for these functions.

Quadratics (2^{nd} degree)

Leading Coefficient	Constant Difference
−1	−2
5	10
−3	−6

Cubics (3^{rd} degree)

Leading Coefficient	Constant Difference
1	6
−3	−18
−4	−24
2	12

Quartics (4^{th} degree)

Leading Coefficient	Constant Difference
1	24
2	48
−3	−72
3	72

Do you see a connection?

Observations

Quadratics (2^{nd} degree)

Leading Coefficient	Constant Difference
-1	-2
5	10
-3	-6
a	$a \times 2$

The value of the constant difference for quadratic functions is twice the value of the leading coefficient.

Cubics (3^{rd} degree)

Leading Coefficient	Constant Difference
1	6
-3	-18
-4	-24
2	12
a	$a \times 6$

For the cubic functions, the constant difference is 6 times the leading coefficient.

Quartics (4^{th} degree)

Leading Coefficient	Constant Difference
1	24
2	48
-3	-72
3	72
a	$a \times 24$

With the quartic functions, the constant difference is 24 times the leading coefficient.

Observations

There is actually a connection between the value of this multiple and the degree of the polynomial.

2^{nd} degree - multiply leading coefficient by 2 $\quad 2 = 2 \times 1$

3^{rd} degree - multiply leading coefficient by 6 $\quad 6 = 3 \times 2 \times 1$

4^{th} degree - multiply leading coefficient by 24 $\quad 24 = 4 \times 3 \times 2 \times 1$

2, 6, and 24 are values that have special significance in math.

The Value of Constant Difference

For a positive integer n, $n!$ (n factorial) is defined by:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

Therefore, $2! = 2$, $3! = 6$, and $4! = 24$.

From our observation, it appears that the n^{th} differences are constant for a polynomial of degree n; the value of the constant difference is given by $\Delta^n y = a \times n!$ where a is the leading coefficient of the function and n is the degree of the polynomial.

This last observation works only if the change in x in the table of values is 1, that is $\Delta x = 1$, which is the case for all the tables presented previously.

The Value of Constant Difference

In actual fact, if $f(x)$ is an n^{th} degree polynomial function, then

$$\frac{(\Delta^n y)}{(\Delta x)^n} = a \times n!$$

where $\Delta^n y$ is the n^{th} constant difference and Δx is the difference in x-values.

So, the n^{th} differences of the polynomial are given by

$$\Delta^n y = a \times n! \times (\Delta x)^n$$

When $\Delta x = 1$, then $\Delta^n y = a \times n!$

The Value of Constant Difference

Example 1

Consider the cubic function $y = -2x^3 + 5x$. Here is a table of values for the function, where the $\Delta x = 2$.

x	$y = -2x^3 + 5x$	Δy	$\Delta^2 y$	$\Delta^3 y$
−3	39	−42	48	−96
−1	−3	6	−48	−96
1	3	−42	−144	−96
3	−39	−186	−240	
5	−225	−426		
7	−651			

This cubic has a constant third difference of -96.

The leading coefficient, a, is -2. The degree of the function is 3. So, $n = 3$ and $\Delta x = 2$.

Notice,

$$\begin{aligned} a \times n! \times (\Delta x)^n &= -2 \times 3! \times 2^3 \\ &= -96 \\ &= \Delta^3 y \end{aligned}$$

The Value of Constant Difference

Example 1

It is very important to note that when we are working with finite differences, the differences in the x values in the table must be constant.

x	$y = -2x^3 + 5x$	Δy	$\Delta^2 y$	$\Delta^3 y$
−3	39	−42	48	−96
−1	−3	6	−48	−96
1	3	−42	−144	−96
3	−39	−186	−240	
5	−225	−426		
7	−651			

Δx	x	$y = -2x^3 + 5x$	Δy	$\Delta^2 y$	$\Delta^3 y$
1	−3	39	−33	30	−36
3	−2	6	−3	−6	−204
1	1	3	−9	−210	−330
3	2	−6	−219	−540	
3	5	−225	−759		
	8	−984			

We can see from this table that a constant third difference is not obtained with the same cubic function used previously.

This is due to the fact that the change in the x values in the table is not constant.

Identifying Polynomial Functions from a Table of Values

Finite differences provide a means for identifying polynomial functions from a table of values.
Knowing the relationship between the value of the constant difference and the leading coefficient of the function can also be useful.

Example 2

Determine the equation of the polynomial function that models the data found in the table.

x	y
-2	-29
-1	-26
0	-5
1	16
2	19
3	-14

Identifying Polynomial Functions from a Table of Values

Example 2

Solution

First, determine the degree of the polynomial function represented by the data by considering finite differences.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-2	-29	3	18	-18
-1	-26	21	0	-18
0	-5	21	-18	-18
1	16	3	-36	
2	19	-33		
3	-14			

Since the third differences are constant, the polynomial function is a cubic.
We can now find the equation using the general cubic function, $y = ax^3 + bx^2 + cx + d$, and determining the values of a, b, c, and d.
We can find the value of the leading coefficient, a, by using our constant difference formula.
From the table, $\Delta x = 1$. The constant difference $\Delta^3 y = -18$.

$$\Delta^3 y = a \times 3! \times (1)^3$$
$$-18 = a \times 3 \times 2 \times 1$$

Solving gives $a = -\frac{18}{6} = -3$.

Identifying Polynomial Functions from a Table of Values

Example 2

Solution

We can now use 3 of the points from the table to create 3 equations and solve for the values of b, c, and d.
A good point to start with is the y-intercept $(0, -5)$ which will provide the value of d.

$$-5 = -3(0)^3 + b(0)^2 + c(0) + d$$

Therefore, $d = -5$.
So far, we have $y = -3x^3 + bx^2 + cx - 5$.

Using $(1, 16)$ gives

$$\begin{aligned} 16 &= -3(1)^3 + b(1)^2 + c(1) - 5 \\ 24 &= b + c \end{aligned}$$

Using $(-1, -26)$ gives

$$\begin{aligned} -26 &= -3(-1)^3 + b(-1)^2 + c(-1) - 5 \\ -24 &= b - c \end{aligned}$$

Identifying Polynomial Functions from a Table of Values

Example 2

Solution

This gives a system of two equations with two unknowns.

$$\begin{aligned} 24 &= b + c \qquad (1) \\ -24 &= b - c \qquad (2) \end{aligned}$$

Adding the two equations, we obtain $2b = 0$ and so, $b = 0$.
Subtracting the two equations, we obtain $2c = 48$ and so, $c = 24$.

Therefore, $y = -3x^3 + 24x - 5$ is the equation of the function.

Summary

- For an n^{th} degree polynomial function, the n^{th} finite differences will be constant if the change in x, Δx, in the table is constant.
- If the change in x is 1 (i.e., $\Delta x = 1$) for a given table of values, then the value of the constant difference, $\Delta^n y$, is $a \times n!$, where a is the leading coefficient and $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$.

  This concept is linked to derivatives, studied in calculus, where the n^{th} order derivative of an n^{th} degree polynomial is constant and the constant value of this derivative is given by $a \times n!$.
- Extension: If $\Delta x = p$, then the constant difference, $\Delta^n y$, will equal $p^n \times a \times n!$.