Introduction to Polynomial Functions

In This Module

- We will review and introduce terminology that will be used throughout the Polynomial Functions unit.
- We will explore the behaviour of the graphs of $y = x^3$ and $y = x^4$.

Polynomials

A **polynomial** is a mathematical expression constructed by the sum and/or difference of algebraic terms. Each term consists of variable factors raised to non-negative integer exponents and multiplied by real numerical coefficients.

Example 1

Consider

$$-\frac{2}{3}x^3y^2z - \sqrt{5}x^2y + \frac{x}{2} + \pi$$

This is a polynomial, although not a typical one.

- The coefficients are all real values $\left(-\frac{2}{3}, -\sqrt{5}, \frac{1}{2}, \pi\right)$.
- The exponents of the variable factors are all non-negative integers. Note that the last term is a constant which is allowed by the definition ($\pi = \pi x^0$).

Polynomials

A **polynomial** is a mathematical expression constructed by the sum and/or difference of algebraic terms. Each term consists of variable factors raised to non-negative integer exponents and multiplied by real numerical coefficients.

Example 2

However,

$$2xy^{-1} + \frac{4}{y^2} + \sqrt{x}$$

is not a polynomial for several reasons.

- The variables have negative exponents $\left(2xy^{-1}, \frac{4}{y^2} = 4y^{-2}\right)$.
- The variables have fractional exponents ($\sqrt{x} = x^{\frac{1}{2}}$).

Degree of Polynomial

The degree of a term of a polynomial is determined by the number of variable factors in the term, and can be calculated by adding the variable exponents in the term.

$$-\frac{2}{3}x^3y^2z - \sqrt{5}x^2y + \frac{x}{2} + \pi$$

Term 1	Term 2	Term 3	Term 4
$-\frac{2}{3}x^3y^2z^1$	$-\sqrt{5}x^2y^1$	$\frac{x^1}{2}$	πx^0
Degree 6	Degree 3	Degree 1	Degree 0

Terms of degree 0 are called **constants**.

Polynomial Functions

A **polynomial function** is a function whose equation is defined by a polynomial in one variable,

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where the numerical coefficients $a_0, a_1, a_2, \ldots, a_n$ are real numbers and the exponents of x given by $n, n-1, n-2, \ldots$ are whole numbers (non-negative integers).

Polynomial Functions

Examples of Polynomial Functions

$$f(x) = x - 2x^2$$

$$y = -\frac{1}{3}x^5 - \sqrt{2}x^3 + 5x + \pi$$

$$g(x) = 3(x-1)^2 - 5$$

In the first two examples, all coefficients are real numbers and the exponents of x are non-negative integers.

We see that $g(x)$ is not in the form of the previous two.
However, by expanding and simplifying the right hand side of the equation, this function can be expressed by

$$g(x) = 3x^2 - 6x - 2$$

Polynomial Functions

Examples of Non-Polynomial Functions

$$h(x) = \frac{1}{x} \text{ or } h(x) = x^{-1}$$

$$y = 2x + |x^3|$$

$$f(x) = 2x^{\frac{1}{3}} - \sqrt{x}$$

$$x = y^2$$

In the first example, exponents of x cannot be negative in value.

Polynomial Functions

Examples of Non-Polynomial Functions

$$h(x) = \frac{1}{x} \text{ or } h(x) = x^{-1}$$

$$y = 2x + |x^3|$$

$$f(x) = 2x^{\frac{1}{3}} - \sqrt{x}$$

$$x = y^2$$

In the second example, $|x^3|$ is not an acceptable term.

Polynomial Functions

Examples of Non-Polynomial Functions

$$h(x) = \frac{1}{x} \text{ or } h(x) = x^{-1}$$

$$y = 2x + |x^3|$$

$$f(x) = 2x^{\frac{1}{3}} - \sqrt{x}$$

$$x = y^2$$

In the third example, exponents of x must be whole numbers.

Polynomial Functions

Examples of Non-Polynomial Functions

$$h(x) = \frac{1}{x} \text{ or } h(x) = x^{-1}$$

$$y = 2x + |x^3|$$

$$f(x) = 2x^{\frac{1}{3}} - \sqrt{x}$$

$$x = y^2$$

The final example is not a function. There are two possible values of y for all positive real values of x.

For example, $x = 25$ gives $y = \pm 5$.

Terminology

The numerical coefficient of the highest degree term in a polynomial is called the **leading coefficient.**

In the general function,

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

a_n is the leading coefficient.
The terms of the polynomial are usually arranged in descending order of the degree of the term.
In this standard form, the leading coefficient is the coefficient of the first term.

In the polynomial function examples,

$$f(x) = x - 2x^2$$
$$y = -\frac{1}{3}x^5 - \sqrt{2}x^3 + 5x + \pi$$
$$g(x) = 3(x-1)^2 - 5$$

the leading coefficients are $-2, -\frac{1}{3}$, and 3, respectively.

Terminology

The domain of all polynomial functions is the set of all real numbers; that is, $D = \{x \mid x \in \mathbb{R}\}$, since there is no restriction on the value of x.
The range of the polynomial function depends on the behaviour of its graph.

The **degree of the polynomial function** is given by the value of the highest exponent of the variable.

Terminology

A **constant function**, $f(x) = a$, is a polynomial function of degree 0 since $f(x) = ax^0$.

A **linear function**, $f(x) = ax + b$, is a polynomial function of degree 1 or less. A constant function is also a linear function.

A **quadratic function**, $f(x) = ax^2 + bx + c, a \neq 0$, is a polynomial function of degree 2.

A 3^{rd} degree polynomial function, $f(x) = ax^3 + bx^2 + cx + d,\ a \neq 0$, is called a **cubic function**.

A 4^{th} degree polynomial function, $f(x) = ax^4 + bx^3 + cx^2 + dx + e,\ a \neq 0$, is called a **quartic function**.

A 5^{th} degree polynomial function, $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f,\ a \neq 0$, is called a **quintic function**.

Any polynomial function with degree n, where $n > 5$, will be referred to as an n^{th} degree polynomial function. Special names exist for some higher degree polynomial functions, but are less commonly used.

Terminology

Even Degree vs. Odd Degree

An **even degree** polynomial is an n^{th} degree polynomial where n is even. Constant (degree 0), quadratic (degree 2), and quartic (degree 4) functions are even degree polynomial functions.

An **odd degree** polynomial is an n^{th} degree polynomial where n is odd. Linear functions of degree 1, cubic (degree 3), and quintic (degree 5) functions are odd degree polynomial functions.

Different Forms of Polynomial Functions

Representing a polynomial in different forms can provide different types of information about the behaviour of the function.

For example, when working with quadratic functions, we have the following:

- In **standard form**, $f(x) = ax^2 + bx + c$, the degree, 2; leading coefficient, a; and y-intercept, c, of the function are easily identified.
- In **factored form**, $y = a(x - p)(x - q)$, the zeros (or x-intercepts), p and q, are readily determined.
- In the form $y = a(x - h)^2 + k$, the vertex of the function, (h, k), can be quickly identified. We can also identify the transformations applied to the parent function, $y = x^2$, which can help us graph the function. For a quadratic function, this form is referred to as **vertex form**.

Different Forms of Polynomial Functions

Example 1

Consider the following polynomial function in factored form:

$$y = -4x(x - 4)(2x + 3)^2$$

To obtain the degree of the polynomial function, we must determine the exponent of the highest degree term of the polynomial.
Expanding to standard form will provide this information:

$$\begin{aligned} y &= -4x(x - 4)(2x + 3)(2x + 3) \\ &= (-4x^2 + 16x)(4x^2 + 12x + 9) \\ &= -16x^4 - 48x^3 - 36x^2 + 64x^3 + 192x^2 + 144x \\ &= -16x^4 + 16x^3 + 156x^2 + 144x \end{aligned}$$

However, simply considering the product of the highest degree terms in each linear factor will give the highest term of the polynomial. In $y = -4x(x - 4)(2x + 3)(2x + 3)$, calculating $-4x(x)(2x)^2$ will also produce $-16x^4$.
Thus, the function is a 4^{th} degree polynomial, or a quartic, and the leading coefficient is -16.

In another situation, we may need to determine the zeros of a polynomial function given to us in standard form. We will need to factor the polynomial. Techniques used to factor polynomials will be a focus of the later half of Polynomial Functions unit.

Simple Polynomial Functions: Power Functions

We will begin our study of polynomial functions by discussing the behaviour of the cubic function, $y = x^3$, and the quartic function, $y = x^4$.

These functions are called **power functions** and are the simplest type of polynomial functions.

Simple Polynomial Functions: Power Functions

A **power function** is a function of the form $f(x) = ax^n$, where $a, n \in \mathbb{R}$.

If the power function is also a polynomial function, then n is an integer and $n \geq 0$.

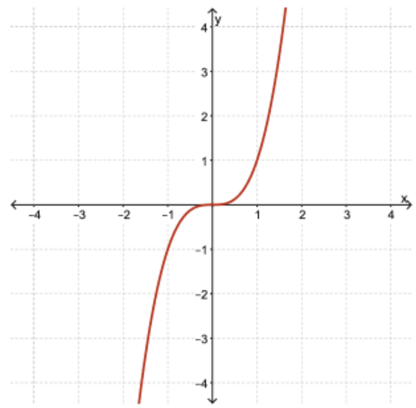
The Cubic Function $y = x^3$

Consider the function $y = x^3$.

Since there are no restrictions on the value of x, the domain of the function is the set of all real numbers (Domain: $\{x \mid x \in \mathbb{R}\}$).

We can now create a table of values for the function and graph the curve.

x	y
-3	-27
-2	-8
-1	-1
$-\frac{1}{2}$	$-\frac{1}{8}$
0	0
$\frac{1}{2}$	$\frac{1}{8}$
1	1
2	8
3	27



Since the function has opposite end behaviour with y approaching both negative and positive infinity, the range of the function is also the set of all real numbers (Range: $\{y \mid y \in \mathbb{R}\}$).

The Cubic Function $y = x^3$

This cubic function's behaviour is somewhat similar to the line $y = x$; it has the same end behaviour and appears to be continuously increasing.

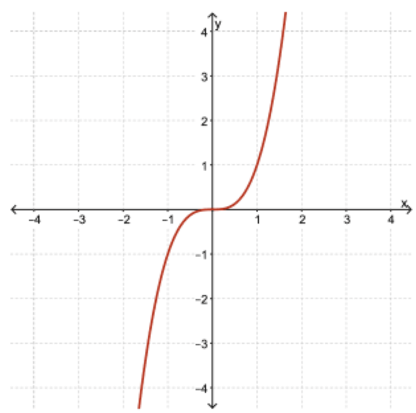
However, the curve of $y = x^3$ seems to linger close to the x-axis as it crosses the axis, causing a slight wave.

This function actually becomes horizontal at $x = 0$.

In this case, the function is said to be **stationary** at $x = 0$ since it is neither increasing nor decreasing at this point.

The shape of the curve also changes at the origin.

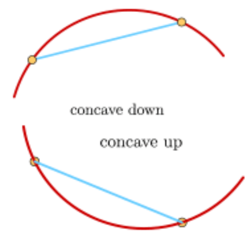
Such a point on a curve is known as a type of **inflection point**.



Inflection Points and Concavity

A **point of inflection** is defined as a point where a graph of a function changes concavity. Concavity is used to describe the way a curve bends.

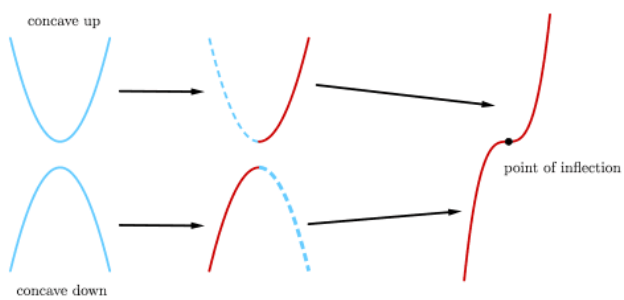
If a line segment joining any two points on a curve is entirely below the curve, then the curve is said to be **concave down** between the two points.
If a line segment is entirely above the curve, the curve is **concave up** between the two points.



Inflection Points and Concavity

Concavity is a topic of study in calculus, but the concept is not difficult to visualize.

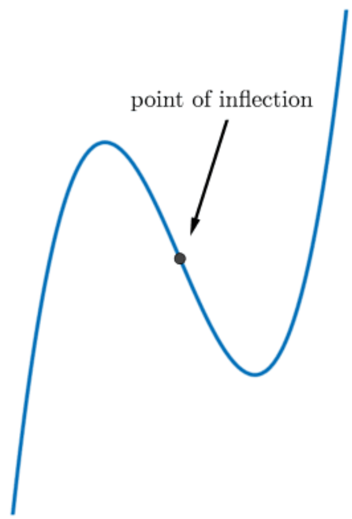
The change in concavity at the point of inflection of $y = x^3$ can be pictured in this way:



We begin with two curves: one entirely concave up the other concave down.
By taking the left branch of the concave down curve and the right branch of the concave up curve and connecting them, we create a curve with an inflection point similar to the one found on the curve of $y = x^3$.

Inflection Points and Concavity

In other situations the change in concavity is more pronounced.
Here the point of inflection occurs between two turning points.

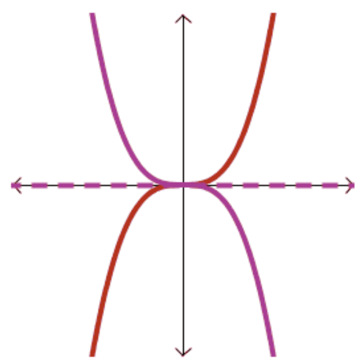


Inflection Points and Concavity

In our study of the graphs of polynomials, we will discuss only those points of inflection similar to the one in the $y = x^3$ function.
In most cases, the location of an inflection point is determined using calculus techniques.

The function $y = x^3$ is an odd function, since it is symmetrical about the origin.
If you reflect the graph of the function in the x-axis and then the y-axis, it will map onto itself.
Algebraically,

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$



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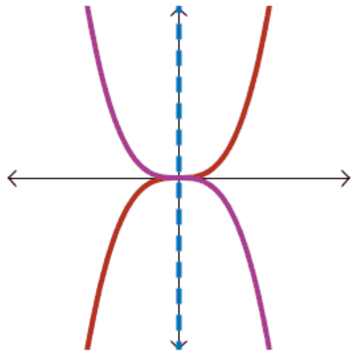
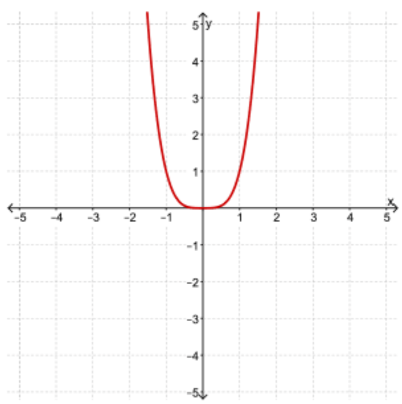


The Quartic Function $y = x^4$

Moving on to the quartic function $y = x^4$, we note that the domain is the set of all real values (Domain: $\{x \mid x \in \mathbb{R}\}$).

We create an appropriate table of values and graph the function.

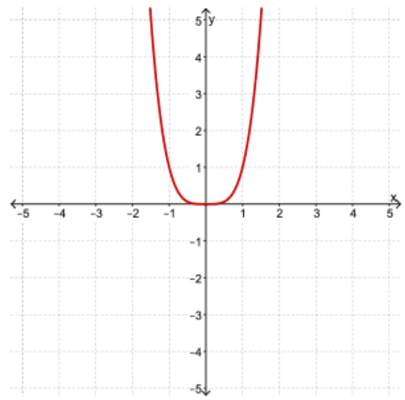
x	y
-3	81
-2	16
-1	1
$-\frac{1}{2}$	$\frac{1}{16}$
0	0
$\frac{1}{2}$	$\frac{1}{16}$
1	1
2	16
3	81



The range can now be identified as the set of all real values greater than or equal to zero (Range: $\{y \mid y \geq 0, y \in \mathbb{R}\}$).

The Quartic Function $y = x^4$

x	y
-3	81
-2	16
-1	1
$-\frac{1}{2}$	$\frac{1}{16}$
0	0
$\frac{1}{2}$	$\frac{1}{16}$
1	1
2	16
3	81



This quartic function's behaviour is similar to that of a quadratic function.
However, the curve lingers closer to the x-axis at the turning point $(0, 0)$ than it does in the quadratic, and this gives the quartic function a broader, flatter appearance.

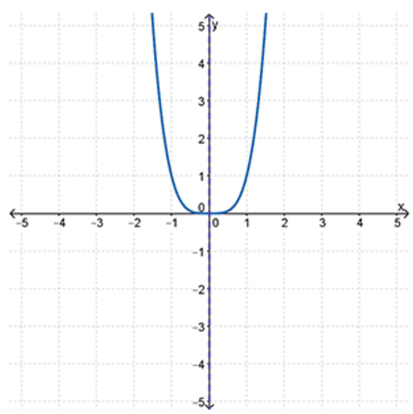
The graph is decreasing when $x < 0$ and increasing when $x > 0$.
The curve is much steeper than a quadratic when $x < -1$ or $x > 1$, as can be seen by the y values in the table.
The graph of this quartic, like the quadratic $y = x^2$, is concave up.

The Quartic Function $y = x^4$

Furthermore, the quartic is both an even degree function $\left(x^4\right)$ and an even function.
That is, it is symmetrical about the y-axis.

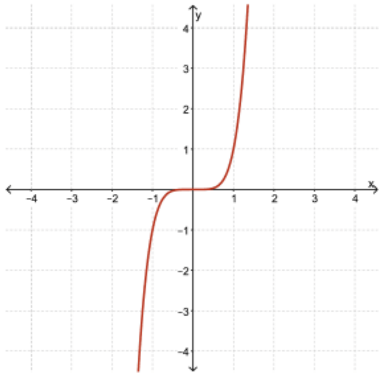
Algebraically,

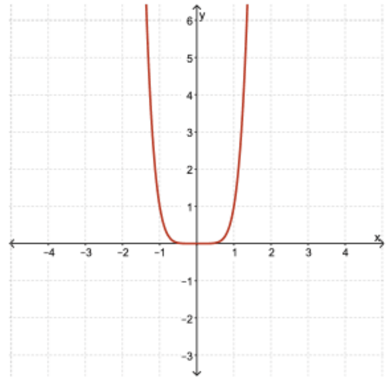
$$f(-x) = (-x)^4 = x^4 = f(x)$$



Higher Degree Power Functions

It is worth noting that the graphs of $y = x^5$ and $y = x^6$ have similar shapes to those of the parent cubic and quartic functions, respectively.





The curves will increase (or decrease) even more quickly when $x < -1$ or $x > 1$, and linger more closely to the x-axis for $-1 < x < 1$.

Higher Degree Power Functions

Considering the value of y when $x = 2$ for each function:

$$2^3 = 8, 2^4 = 16, 2^5 = 32, 2^6 = 64, \ldots$$

we see that the y value grows much quicker as the exponent of the power function is increased.

When $x = \frac{1}{2}$:

$$\left(\frac{1}{2}\right)^3 = \frac{1}{8}, \left(\frac{1}{2}\right)^4 = \frac{1}{16}, \left(\frac{1}{2}\right)^5 = \frac{1}{32}, \left(\frac{1}{2}\right)^6 = \frac{1}{64}, \cdots$$

we can observe the behaviour of the flattening out as the degree of the power function increases.

In general, this behaviour repeats itself for $y = x^n$.
The function will have the shape of $y = x^3$ with a point of inflection if n is odd and it will have the shape of $y = x^2$ with a turning point if n is even.

Summary

In this module, we have defined polynomial functions and discussed some basic characteristics and terminology that apply to these functions.

In the subsequent modules, we will explore the affects of transformations on the cubic and quartic power functions, as you did previously with the quadratic power function, $y = x^2$, and explore the behaviour of more general polynomial functions.

As an extension to our knowledge of power functions, you may wish to investigate the behaviour of functions of the form $y = x^n$ where n is not an integer, but a positive rational value.

For example, consider $y = x^{\frac{1}{2}}$ $(y = \sqrt{x})$, $y = x^{\frac{1}{3}}$, $y = x^{\frac{1}{4}}, \cdots$

You could extend this to consider functions such as:

$$y = x^{\frac{2}{3}}, y = x^{\frac{3}{4}}, y = x^{\frac{5}{4}}, y = x^{\frac{2}{5}}, y = x^{\frac{3}{5}}, y = x^{\frac{4}{5}}, \cdots$$

These functions are not polynomial functions and are not a focus in the high school curriculum.

However, their graphs are not difficult to determine using the knowledge you have of functions and exponents.

When you compare the graphs, some interesting patterns will emerge.