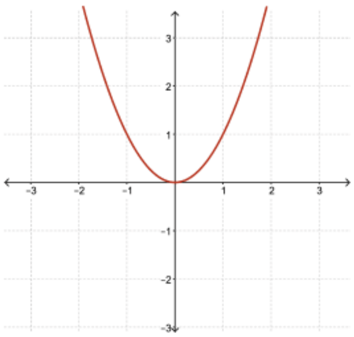
Transformations Of Simple Polynomial Functions

Quadratic Functions

Parent Quadratic Function: $y = x^2$



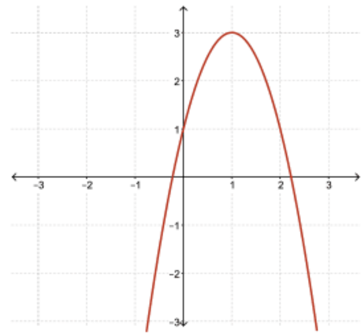
Domain: $\{x \mid x \in \mathbb{R}\}$

Range: $\{y \mid y \geq 0, y \in \mathbb{R}\}$

$$y = ax^2 + bx + c$$

→

Function: $y = -2(x-1)^2 + 3$



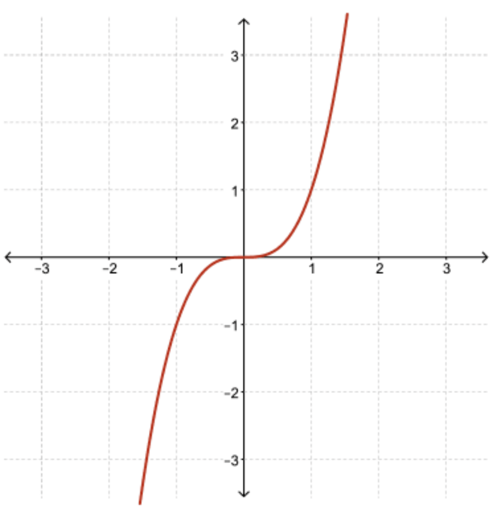
Domain: $\{x \mid x \in \mathbb{R}\}$

Range: $\{y \mid y \leq 3, y \in \mathbb{R}\}$

$$y = a(x-h)^2 + k$$

Cubic and Quartic Functions

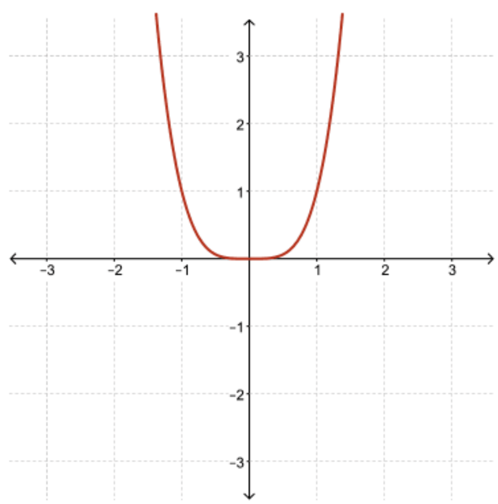
Parent Cubic Function: $y = x^3$



Domain: $\{x \mid x \in \mathbb{R}\}$

Range: $\{y \mid y \in \mathbb{R}\}$

Parent Quartic Functions: $y = x^4$



Domain: $\{x \mid x \in \mathbb{R}\}$

Range: $\{y \mid y \geq 0, y \in \mathbb{R}\}$

Transformational Form

Transformations on a function $y = f(x)$ can be identified when the function is in the form

$$y = af(b(x - h)) + k$$

Cubic $f(x) = x^3$

$$y = a(b(x - h))^3 + k$$

Quartic $f(x) = x^4$

$$y = a(b(x - h))^4 + k$$

Let's review the role of the parameters a, b, h, and k in transforming these functions.

Transformational Form

Cubic $f(x) = x^3$

$$y = a(b(x - h))^3 + k$$

Quartic $f(x) = x^4$

$$y = a(b(x - h))^4 + k$$

Let's review the role of the parameters a, b, h, and k in transforming these functions.

$$y = af(b(x - h)) + k$$

Effect of a:

- If $a < 0$, $f(x)$ is reflected in the x-axis.
- $f(x)$ is stretched vertically from the x-axis by a factor of $|a|$.

Effect of b:

- If $b < 0$, $f(x)$ is reflected in the y-axis.
- $f(x)$ is stretched horizontally from the y-axis by a factor of $\frac{1}{|b|}$.

Effect of h:

- $f(x)$ is translated horizontally h units.

Effect of k:

- $f(x)$ is translated vertically k units.

Transformational Form

Summary

The transformation of each point is defined by the mapping $(x, y) \rightarrow \left(\frac{1}{b}x + h, ay + k\right)$.
When applying the transformations to the graph of the function, the stretches and/or reflections must be performed first (in any order) prior to the translations.

Applying Transformations

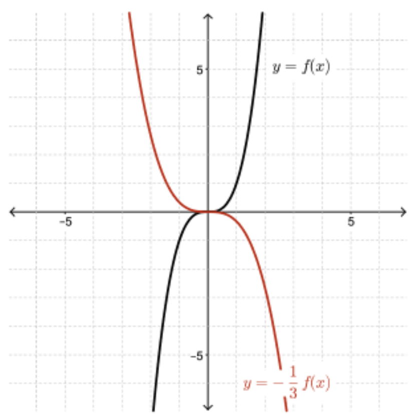
Example 1

Describe the transformations applied to $y = x^3$ to obtain the graph $y = -\frac{1}{3}(x+2)^3 - 1$ and graph the function.

Solution

This function is the image of $f(x) = x^3$ under the transformations defined by $y = -\frac{1}{3}f(x+2) - 1$.

- Since $a = -\frac{1}{3}$, the cubic function is reflected in the x-axis ($a < 0$) and vertically stretched from the x-axis by a factor of $\frac{1}{3}$.
  The image of each point on the graph can be found by multiplying the y-coordinate of the point by $-\frac{1}{3}$ and leaving the x-coordinate the same.
  The graph of this function appears compressed, since the stretch factor is between 0 and 1.
- Since $b = 1$, no horizontal stretch or reflection is applied.



Applying Transformations

Example 1

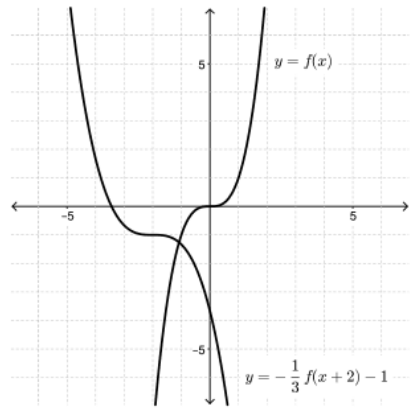
Describe the transformations applied to $y = x^3$ to obtain the graph $y = -\frac{1}{3}(x+2)^3 - 1$ and graph the function.

Solution

This function is the image of $f(x) = x^3$ under the transformations defined by $y = -\frac{1}{3}f(x+2) - 1$.

- Since $h = -2$ and $k = -1$, the function is then translated 2 units left and 1 unit down.

The image graph can be obtained by applying the mapping $(x, y) \to \left(x - 2, -\frac{1}{3}y - 1\right)$ to the original points on the curve.



Applying Transformations

Example 2

Describe the transformations applied to $y = x^4$ to obtain the graph $y = (-2x+4)^4 - 5$. Graph the function and state the domain and range.

Solution

This is a transformation of the quartic function $f(x) = x^4$ under the transformations defined by $y = f(-2(x-2)) - 5$.
Note that the coefficient of x, -2, must be factored from the two terms in the brackets to be in standard transformational form and to help us identify the correct horizontal translation.

Applying Transformations

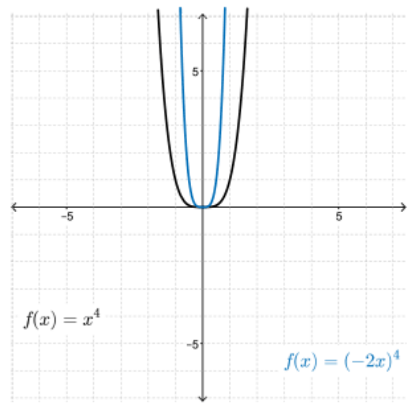
Example 2

Describe the transformations applied to $y = x^4$ to obtain the graph $y = (-2x + 4)^4 - 5$. Graph the function and state the domain and range.

Solution

The equation becomes $y = (-2(x - 2))^4 - 5$.
The transformations being applied to $y = x^4$, in order, would be

- a reflection in the y-axis ($b < 0$),
- a horizontal stretch from the y-axis by a factor of $\frac{1}{2}$ ($|b| = 2$),



Applying Transformations

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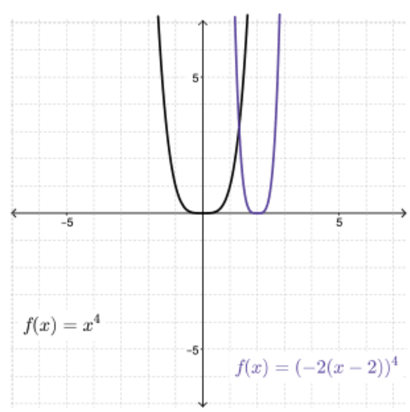
- a reflection in the y-axis ($b < 0$),
- a horizontal stretch from the y-axis by a factor of $\frac{1}{2}$ ($|b| = 2$),
- a horizontal translation to the right 2 units ($h = 2$), and



Applying Transformations

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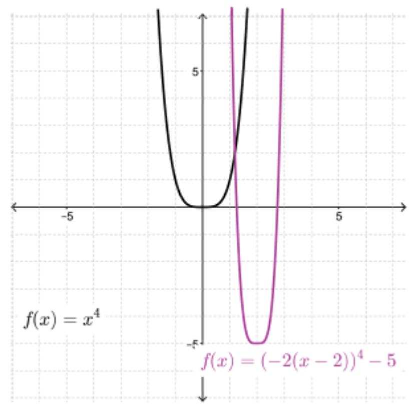
Describe the transformations applied to $y = x^4$ to obtain the graph $y = (-2x + 4)^4 - 5$. Graph the function and state the domain and range.

Solution

The equation becomes $y = (-2(x - 2))^4 - 5$.
The transformations being applied to $y = x^4$, in order, would be

- a reflection in the y-axis ($b < 0$),
- a horizontal stretch from the y-axis by a factor of $\frac{1}{2}$ ($|b| = 2$),
- a horizontal translation to the right 2 units ($h = 2$), and
- a vertical translation down 5 units ($k = -5$).



Applying Transformations

Example 2

Describe the transformations applied to $y = x^4$ to obtain the graph $y = (-2x + 4)^4 - 5$. Graph the function and state the domain and range.

Solution

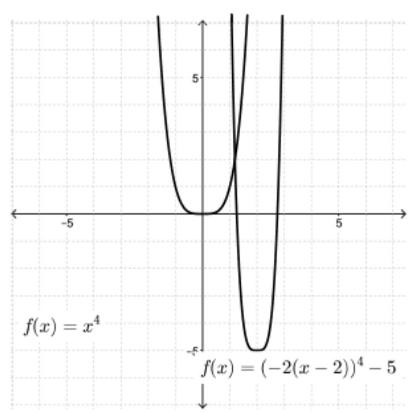
The equation becomes $y = (-2(x - 2))^4 - 5$.
The transformations being applied to $y = x^4$, in order, would be

- a reflection in the y-axis ($b < 0$),
- a horizontal stretch from the y-axis by a factor of $\frac{1}{2}$ ($|b| = 2$),
- a horizontal translation to the right 2 units ($h = 2$), and
- a vertical translation down 5 units ($k = -5$).

The points on the new function can be obtained using the mapping

$$(x, y) \rightarrow \left(-\frac{1}{2}x + 2, y - 5\right)$$

The domain is $\{x \mid x \in \mathbb{R}\}$ and the range is $\{y \mid y \geq -5, y \in \mathbb{R}\}$.



Applying Transformations

Example 2

Describe the transformations applied to $y = x^4$ to obtain the graph $y = (-2x + 4)^4 - 5$. Graph the function and state the domain and range.

Solution

Another approach to graphing the function involves simplifying the equation first.

$$\begin{aligned} y &= (-2(x-2))^4 - 5 \\ &= (-2)^4(x-2)^4 - 5 \\ &= 16(x-2)^4 - 5 \end{aligned}$$

This new form of the equation changes the transformations applied.
It eliminates the reflection in the y-axis, which does nothing to change the graph of $y = x^4$, and the horizontal stretch by a factor of $\frac{1}{2}$ becomes a vertical stretch by a factor of 16.
This new mapping would be defined by

$$(x, y) \rightarrow \left(x + 2, 16y - 5\right)$$

Applying Transformations

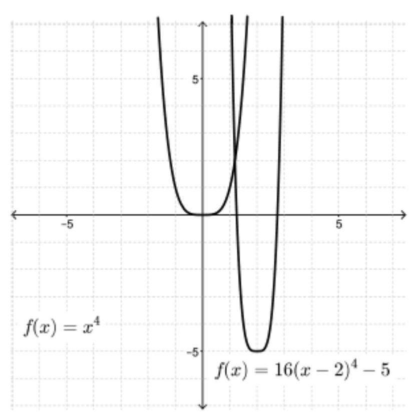
Example 2

Describe the transformations applied to $y = x^4$ to obtain the graph $y = (-2x + 4)^4 - 5$. Graph the function and state the domain and range.

Solution

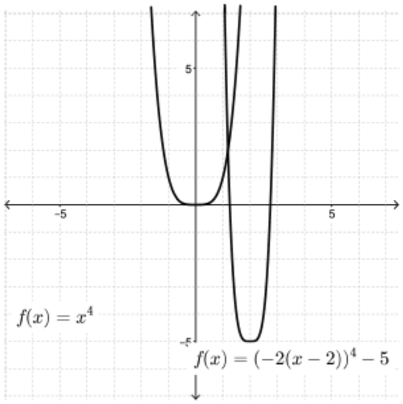
The equation becomes $y = 16(x - 2)^4 - 5$.
The transformations being applied to $y = x^4$, in order, would be

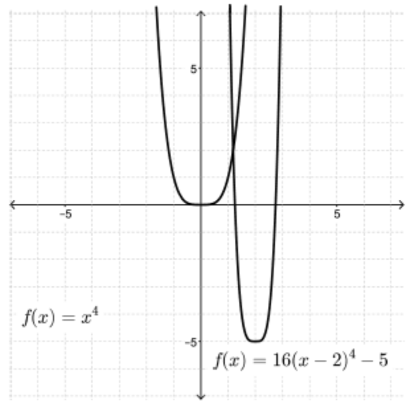
- a vertical stretch from the x-axis by a factor of 16 ($|a| = 16$),
- a horizontal translation to the right 2 units ($h = 2$), and
- a vertical translation down 5 units ($k = -5$).



Equivalence of Methods

We can see by the respective image graphs that the transformations used in either method resulted in the same final graph.



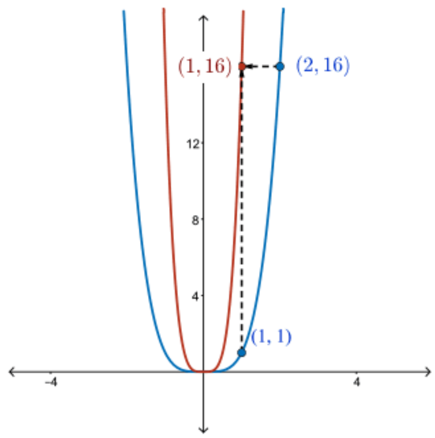


Mathematically, $y = (-2(x-2))^4 - 5$ is equivalent to $y = 16(x-2)^4 - 5$.
In this example, the reflection in the y-axis and the horizontal stretch by a factor of $\frac{1}{2}$ in the first transformation was combined into a single vertical stretch by 16. The translations will not change.
As mentioned earlier, the reflection in the y-axis does nothing to change the graph since the function is symmetric about the y-axis.

Equivalence of Methods

Considering just the stretches under each mapping, we see the equivalence of both transformations.

$(x, y) \rightarrow \left(\frac{1}{2}x, y\right)$	$(x, y) \rightarrow (x, 16y)$
$(2, 16) \rightarrow (1, 16)$	$(1, 1) \rightarrow (1, 16)$
Similarly,	Similarly,
$(-2, 16) \rightarrow (-1, 16)$	$(-1, 1) \rightarrow (-1, 16)$
$(4, 256) \rightarrow (2, 256)$	$(2, 16) \rightarrow (2, 256)$
$(-4, 256) \rightarrow (-2, 256)$	$(-2, 16) \rightarrow (-2, 256)$



The translation to the right by 2 units and down by 5 units is common to both methods, so both will produce the same graph of the transformed function.

Equivalence of Methods

When working with parent polynomial functions of the form $y = x^n$, $n \in \mathbb{Z}$, $n > 0$, it is worth noting that the parameters, a and b, responsible for stretches and reflections can be combined to create a single parameter responsible for the vertical stretch and reflection in the x-axis, by simplifying the equation algebraically.

Consider a general polynomial function of this form:

$$y = a(b(x-h))^n + k, \text{ which is the same as}$$

$$y = ab^n(x-h)^n + k$$

The coefficient ab^n is responsible for the vertical stretch and reflection in the x-axis. It replaces the original horizontal stretch and reflection by b and vertical stretch and reflection by a. The translations remain unchanged. This knowledge can be used to simplify the process of graphing.

Equivalence of Methods

For example,

$$y = \frac{1}{2}(-2x+8)^3 - 5$$

or

$$y = \frac{1}{2}(-2(x-4))^3 - 5$$

The transformations applied to $y = x^3$ include a vertical stretch from the x-axis by a factor of $\frac{1}{2}$, a reflection in the y-axis, a horizontal stretch from the y-axis by a factor of $\frac{1}{2}$, as well as a horizontal translation 4 units right and a vertical translation 5 units down.

Note that the equation can be simplified to

$$y = \frac{1}{2}(-2)^3(x-4)^3 - 5$$

or

$$y = -4(x-4)^3 - 5$$

The stretches and reflections are replaced by a reflection in the x-axis and a vertical stretch by a factor of 4. The translations remain the same.

The graph of the image function will be the same regardless of which combination of transformations is applied.

Equations of Transformed Functions

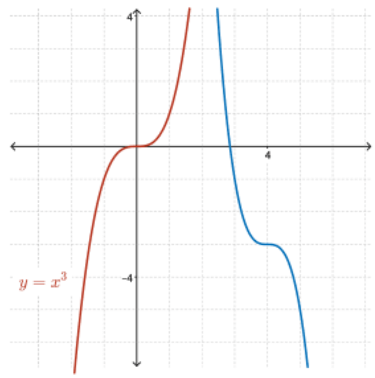
Example 3

Transformations are applied to the cubic function, $y = x^3$, to obtain the resulting graph (in blue). Determine the equation for the transformed function.

Solution

We will determine the equation of the transformed function using the general form $y = a(x - h)^3 + k$. By identifying the specific transformations applied, we can determine the values of a, h, and k.
We need not consider any horizontal stretch or reflection in the y-axis.
The point of inflection $(0, 0)$ on the original function, $y = x^3$, is not affected by any reflection and/or stretch since the general mapping $(x, y) \rightarrow \left(x, ay\right)$ covers all reflections and stretches. Thus, $(0, 0) \rightarrow (0, a(0)) \rightarrow (0, 0)$.
The point $(0, 0)$ on the parent function has been mapped to $(4, -3)$.
This tells us that a horizontal translation right 4 units ($h = 4$) and a vertical translation down 3 units ($k = -3$) has been applied.

$$y = a(x - 4)^3 - 3$$



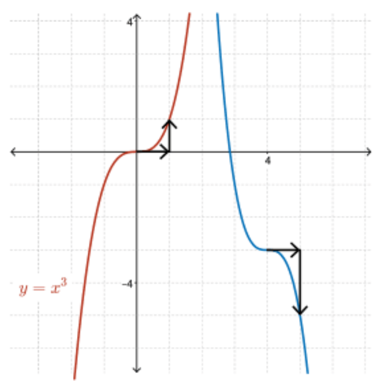
Equations of Transformed Functions

Example 3

Transformations are applied to the cubic function, $y = x^3$, to obtain the resulting graph (in blue). Determine the equation for the transformed function.

Solution

The graph has been reflected in either the x-axis or the y-axis (equivalent in the case of cubic functions which are symmetrical about the origin).
We will work with a reflection in the x-axis; therefore, a is negative.
If we move 1 unit right of the point of inflection on the base graph, the appropriate point on the curve is 1 unit up.
On the transformed graph, the corresponding point is 2 units down.
This would happen if there was a vertical stretch by a factor of 2 applied after a reflection ($a = -2$).
Therefore, the equation of the transformed function would be $y = -2(x - 4)^3 - 3$.



Equations of Transformed Functions

Example 3

Transformations are applied to the cubic function, $y = x^3$, to obtain the resulting graph (in blue). Determine the equation for the transformed function.

Solution

The stretch factor can also be determined algebraically.
If we substitute $h = 4$ and $k = -3$ into the general form, $y = a(x - h)^3 + k$, we have

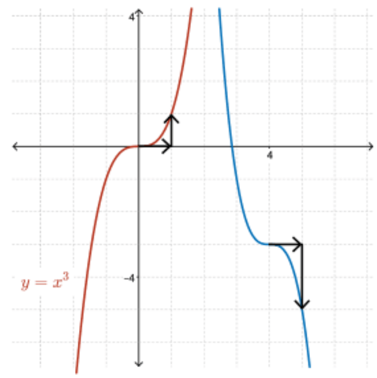
$$y = a(x - 4)^3 - 3$$

Substituting a point on the curve other than $(4, -3)$, the value of a can be determined. Using the point $(3, -1)$,

$$-1 = a(3 - 4)^3 - 3$$
$$-1 = -a - 3$$
$$a = -2$$

So,

$$y = -2(x - 4)^3 - 3$$



Equations of Transformed Functions

Example 4

The graph of the function $f(x) = ax^n + k$, $a \neq 0$, $n \in \mathbb{N}$, is stretched vertically by a factor of 3, reflected in the y-axis, horizontally stretched by a factor of 2, shifted left 2 units and up 5.
Three points on this new graph are $\left(-8, \frac{133}{2}\right)$, $(-2, -55)$, and $(2, 31)$.
Determine the equation of $y = f(x)$ and the equation of the new function under the transformations applied.

Solution

Let $y = g(x)$ be the image of $y = f(x)$ under the given transformations.
That is, $g(x) = 3f\left(-\frac{1}{2}(x + 2)\right) + 5$.
Let (x, y) be a point on $y = f(x)$; then $(-2x - 2, 3y + 5)$ would be the image of the point on $y = g(x)$.
The point $\left(-8, \frac{133}{2}\right)$ is on $y = g(x)$. Therefore, $-8 = -2x - 2$ and $\frac{133}{2} = 3y + 5$.
Solving both equations for x and y, respectively, gives $x = 3$ and $y = \frac{41}{2}$.
Hence, $\left(3, \frac{41}{2}\right)$ is the corresponding point on $y = f(x)$.
Similarly, using $(-2, 55)$, we obtain $-2 = -2x - 2$ and $-55 = 3y + 5$. Therefore, $(0, -20)$ is a second point on $y = f(x)$.
Using $(2, -31)$, we have $2 = -2x - 2$ and $-31 = 3y + 5$. Therefore, $(-2, -12)$ is a third point on the graph $y = f(x)$.

Equations of Transformed Functions

Example 4

The graph of the function $f(x) = ax^n + k$, $a \neq 0$, $n \in \mathbb{N}$, is stretched vertically by a factor of 3, reflected in the y-axis, horizontally stretched by a factor of 2, shifted left 2 units and up 5.
Three points on this new graph are $\left(-8, \frac{133}{2}\right)$, $(-2, -55)$, and $(2, 31)$.
Determine the equation of $y = f(x)$ and the equation of the new function under the transformations applied.

Solution

To determine the equation of $f(x)$, where $f(x) = ax^n + k$, we must solve for the three unknowns: a, n, and k.
Note that the three points $\left(3, \frac{41}{2}\right)$, $(0, -20)$, and $(-2, -12)$ on the graph of the function must satisfy the equation $f(x) = ax^n + k$.

$$\frac{41}{2} = a(3)^n + k \qquad (1)$$
$$-20 = a(0)^n + k \qquad (2)$$
$$-12 = a(-2)^n + k \qquad (3)$$

Equations of Transformed Functions

Example 4

The graph of the function $f(x) = ax^n + k$, $a \neq 0$, $n \in \mathbb{N}$, is stretched vertically by a factor of 3, reflected in the y-axis, horizontally stretched by a factor of 2, shifted left 2 units and up 5.
Three points on this new graph are $\left(-8, \frac{133}{2}\right)$, $(-2, -55)$, and $(2, 31)$.
Determine the equation of $y = f(x)$ and the equation of the new function under the transformations applied.

Solution

$$\frac{41}{2} = a(3)^n + k \qquad (1)$$
$$-20 = a(0)^n + k \qquad (2)$$
$$-12 = a(-2)^n + k \qquad (3)$$

From (2), $k = -20$. Substituting this into (1) and (3) gives

$$\frac{41}{2} = a(3)^n - 20$$
$$\frac{41}{2} + 20 = a(3)^n$$
$$a(3)^n = \frac{81}{2} \qquad (4)$$

$$-12 = a(-2)^n - 20$$
$$a(-2)^n = -12 + 20$$
$$a(-2)^n = 8 \qquad (5)$$

Equations of Transformed Functions

Example 4

The graph of the function $f(x) = ax^n + k$, $a \neq 0$, $n \in \mathbb{N}$, is stretched vertically by a factor of 3, reflected in the y-axis, horizontally stretched by a factor of 2, shifted left 2 units and up 5.
Three points on this new graph are $\left(-8, \frac{133}{2}\right)$, $(-2, -55)$, and $(2, 31)$.
Determine the equation of $y = f(x)$ and the equation of the new function under the transformations applied.

Solution

$$a(3)^n = \frac{81}{2} \qquad (4)$$

$$a(-2)^n = 8 \qquad (5)$$

Dividing (4) by (5)

$$\frac{a(3)^n}{a(-2)^n} = \frac{\frac{81}{2}}{8}$$

$$\left(-\frac{3}{2}\right)^n = \frac{81}{16}$$

Therefore, $n = 4$. Substituting into (5), we see $a(-2)^4 = 8$, so $a = \frac{1}{2}$.

Equations of Transformed Functions

Example 4

The graph of the function $f(x) = ax^n + k$, $a \neq 0$, $n \in \mathbb{N}$, is stretched vertically by a factor of 3, reflected in the y-axis, horizontally stretched by a factor of 2, shifted left 2 units and up 5.
Three points on this new graph are $\left(-8, \frac{133}{2}\right)$, $(-2, -55)$, and $(2, 31)$.
Determine the equation of $y = f(x)$ and the equation of the new function under the transformations applied.

Solution

Therefore, the equation of the original function is $f(x) = \frac{1}{2}x^4 - 20$. The equation of the image function, $g(x)$, is

Summary

- The function, $y = x^n, n \in \mathbb{Z}, n > 1$, will have a shape similar to $y = x^2$ with a turning point at the origin when n is even, and a shape similar to $y = x^3$ with a point of inflection at the origin when n is odd.
- The graph of the function, $y = a[b(x - h)]^n + k$, is obtained by applying, in order, the following transformations to the parent function $y = x^n$:

  a vertical stretch from the x-axis by a factor of $|a|$ and if $a < 0$, a reflection in the x-axis,

  a horizontal stretch from the y-axis by a factor of $\frac{1}{|b|}$ and if $b < 0$, a reflection in the y-axis,

  a horizontal translation h unit(s), and

  a vertical translation k unit(s).
- Each point, (x, y), on the parent function will map to a point, $\left(\frac{1}{b}x + h, ay + k\right)$, on the transformed function.
- The equation of the function, $y = a[b(x + h)]^n + k$, can be simplified to $y = c(x - h)^n + k$, where $c = ab^n$, thus eliminating the reflection in the y-axis and horizontal stretch and reducing the mapping to $(x, y) \rightarrow (x + h, cy + k)$.
- The domain of the function, $y = a[b(x + h)]^n + k$, is $\{x \mid x \in \mathbb{R}\}$. The range of the function, $y = a[b(x + h)]^n + k$, is

  $\{y \mid y \in \mathbb{R}\}$ when n is odd,

  $\{y \mid y \leq k, y \in \mathbb{R}\}$ when n is even and $a < 0$, and

  $\{y \mid y \geq k, y \in \mathbb{R}\}$ when n is even and $a > 0$.