The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Determining Average and Approximating Instantaneous Rates of Change for Linear, Polynomial, and Rational Functions

Introduction

In this module, we will examine average rate of change and instantaneous rate of change, making connections to the slope of secants and tangents.

A **rate of change** is a measure of the change in the dependent variable, Δy, with respect to a change in the independent variable, Δx. When we calculate the slope of a line segment, we are calculating the rate of change of y with respect to x.

We are interested in examining two types of rates of change.
The first is average rate of change which is measured over an interval.
The second is instantaneous rate of change which is measured at a particular instant.

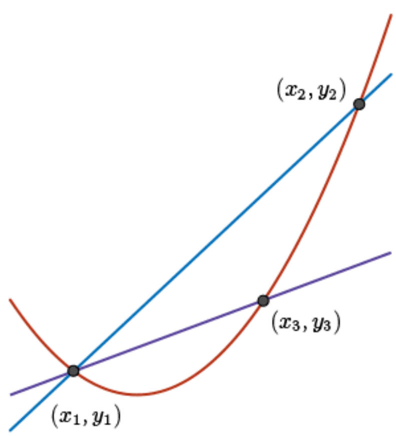
Secants and Tangents

A **secant** is a line which passes through a curve in at least two distinct points.

More than one secant can be drawn through a point on a curve.

A secant through a single point is not unique.
The **slope of the secant** between two points represents the **average rate of change** and can be found as follows:

$$\begin{aligned} m_{\text{secant}} &= \text{average rate of change} \\ &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \end{aligned}$$



The notation m_{secant} is read "m subscript secant" and is used to represent the slope of the secant.
For something like m_{PQ}, we read "m subscript PQ." This notation denotes the slope of a secant through PQ.

Secants and Tangents

A **tangent** is a line which touches a curve at a point.

The point is called a point of tangency. At the point of tangency, the tangent (line) does not cross the curve but it may or may not cross the curve at some other point.

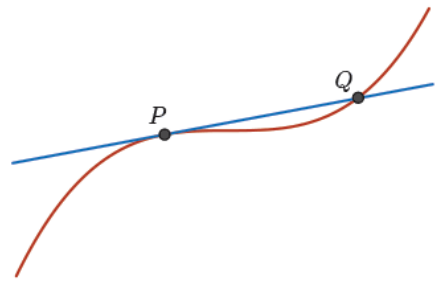
On the diagram, the line is tangent to the curve at point P and crosses the curve at point Q.

The line is not tangent to the curve at point Q.

A tangent most resembles the curve near that point.

The slope of the tangent to a specific point on the curve represents the instantaneous rate of change of the curve at that point.

This idea will be pursued later in the module.

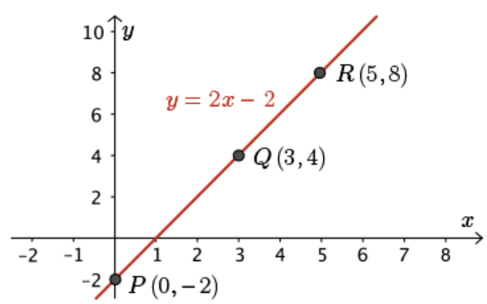


Rates of Change for Linear Functions

Consider the linear function $y = 2x - 2$.

A graph of $y = 2x - 2$ is shown with three specific points $P(0, -2)$, $Q(3, 4)$, and $R(5, 8)$ also plotted on the graph.

$$m_{\text{PQ}} = \frac{4+2}{3-0} = 2$$
$$m_{\text{PR}} = \frac{8+2}{5-0} = 2$$
$$m_{\text{QR}} = \frac{8-4}{5-3} = 2$$



Again, the notation m_{PQ} is read "m subscript PQ" and is used to represent the slope of the secant through PQ.

The slope of every line segment contained on the line $y = 2x - 2$ is the same as the slope of the line.

In fact, the average rate of change for any two points on a linear function is constant.

The average rate of change for a linear function is the same as the slope of the line segment which, in turn, is the same as the slope of the linear function.

It follows that the instantaneous rate of change at any point on the linear function is also the same as the average rate of change between any two points on the line.

However, finding the average rate of change and the instantaneous rate of change of a curve presents a different challenge: the slope is not constant at every point.

Golf Anyone?

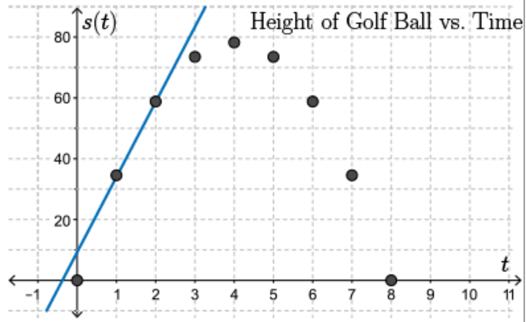
On a distance-time graph, the slope of the secant (or average rate of change) represents the average velocity.

$$m_{\text{secant}} = \text{average rate of change} = \text{average velocity} = \frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1}$$

Consider data for the trajectory of a golf ball being struck off the ground.

The table summarizes the height of the ball (measured in metres) with respect to time (measured in seconds).

Time t (seconds)	Height $s(t)$ (metres)
0	0
1	34.3
2	58.8
3	73.5
4	78.4
5	73.5
6	58.8
7	34.3
8	0



A secant which passes through $(1, 34.3)$ and $(2, 58.8)$ is shown on the graph of the data.

Algebraically, the average rate of change or average velocity between 1 and 2 seconds can be calculated by finding the slope of the secant between the points $(1, 34.3)$ and $(2, 58.8)$.

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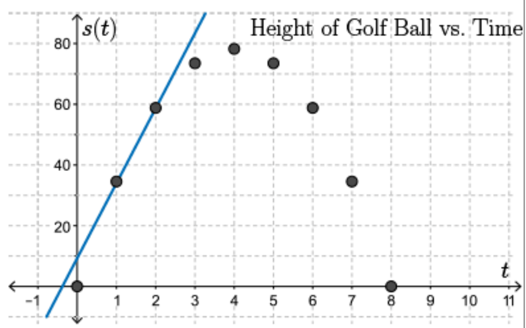


$$m_{\text{secant}} = \text{average velocity} = \frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta s}{\Delta t} = \frac{58.8 - 34.3}{2 - 1} = 24.5 \text{ m/s}$$

Examples

Example 1

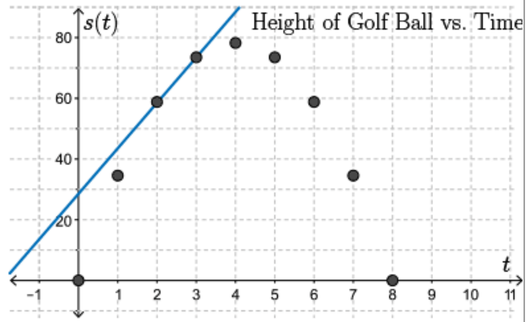
The table contains data relating the height of a golf ball, $s(t)$ in metres, with respect to the time, t in seconds, after the ball was struck.

Time t (seconds)	0	1	2	3	4	5	6	7	8
Height $s(t)$ (metres)	0	34.3	58.8	73.5	78.4	73.5	58.8	34.3	0

a. Sketch the graph and then draw the secant between the points located at 2 and 3 seconds.

Solution

A secant through $(2, 58.8)$ and $(3, 73.5)$ is drawn on the graph.



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The table contains data relating the height of a golf ball, $s(t)$ in metres, with respect to the time, t in seconds, after the ball was struck.

Time t (seconds)	0	1	2	3	4	5	6	7	8
Height $s(t)$ (metres)	0	34.3	58.8	73.5	78.4	73.5	58.8	34.3	0

b. Calculate the average rate of change, or the average velocity, between 2 and 3 seconds.

Solution

Using the points $(2, 58.8)$ and $(3, 73.5)$, we can calculate the slope of the secant.

This is the average rate of change (or average velocity) between these two points.

$$m_{\text{secant}} = \text{average velocity} = \frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta s}{\Delta t} = \frac{73.5 - 58.8}{3 - 2} = 14.7 \text{ m/s}$$

This average velocity is different from the average velocity between 1 and 2 seconds that was calculated earlier.

Modelling the Data Algebraically

Now suppose that the height of the golf ball at any time, t in seconds, is given by the formula $s(t) = -4.9t^2 + 39.2t$, where $s(t)$ is measured in metres.

For example, the height after 2 seconds is equal to $s(2) = -4.9(2)^2 + 39.2(2) = 58.8$ m.

This is the value we had in our table when $t = 2$ seconds.

Using this algebraic model, we are able to find the average velocity over any time interval.

Modelling the Data Algebraically

In particular, we can find the average velocity over the interval from $t = 3$ to $t = 3.5$.

The height of the golf ball at $t = 3$ seconds is $s(3) = 73.5$ m and the height of the golf ball at $t = 3.5$ seconds is $s(3.5) = 77.175$ m.

We can now calculate the average velocity over this time interval as follows:

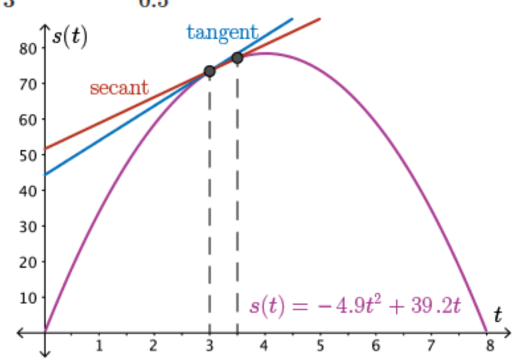
$$\text{average velocity} = \frac{\Delta s}{\Delta t} = \frac{s(3.5) - s(3)}{3.5 - 3} = \frac{77.175 - 73.5}{0.5} = 7.35 \text{ m/s}$$

Notice that we picked a reasonably small time interval: $\Delta t = 3.5 - 3 = 0.5$ seconds.

The sketch shows the function $s(t) = -4.9t^2 + 39.2t$.

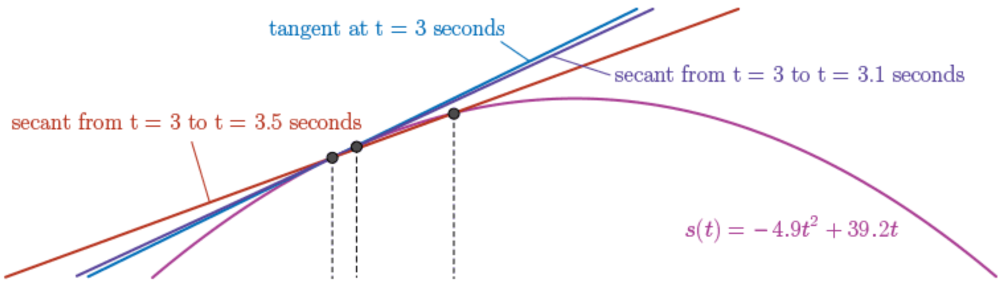
The tangent to the point $(3, 73.5)$ is shown on the graph.

The sketch also includes a secant drawn for the interval from $t = 3$ to $t = 3.5$ seconds.



Approximating the Slope of the Tangent

If we draw a secant for the time interval from $t = 3$ to $t = 3.1$ seconds, the slope of the secant will more closely resemble the slope of the tangent at $t = 3$ seconds.

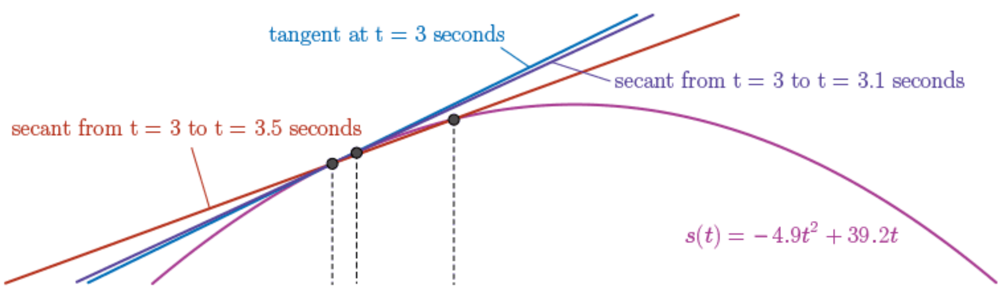


Using the time interval from 3 to 3.1 seconds gives $\Delta t = 3.1 - 3 = 0.1$.
Since $s(3.1) = -4.9(3.1)^2 + 39.2(3.1) = 74.431$ m,

$$m_{\text{secant}} = \text{average velocity} = \frac{\Delta s}{\Delta t} = \frac{s(3.1) - s(3)}{3.1 - 3} = \frac{74.431 - 73.5}{0.1} = 9.31 \text{ m/s}$$

Approximating the Slope of the Tangent

If we draw a secant for the time interval from $t = 3$ to $t = 3.1$ seconds, the slope of the secant will more closely resemble the slope of the tangent at $t = 3$ seconds.

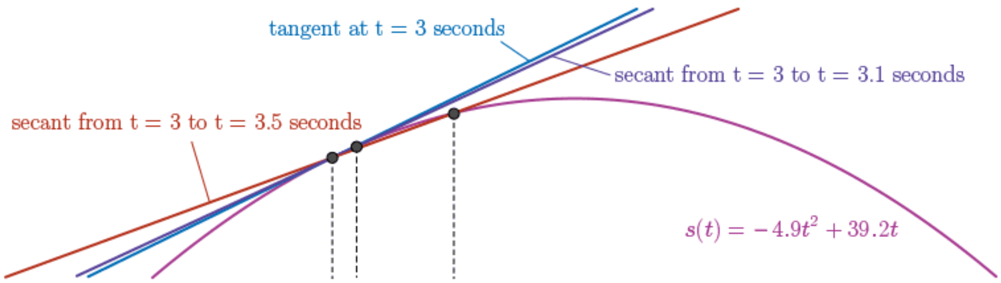


Now, let's try it with a smaller time interval from 3 to 3.01 seconds.
This gives $\Delta t = 3.01 - 3 = 0.01$.
Since $s(3.01) = -4.9(3.01)^2 + 39.2(3.01) = 73.59751$ m,

$$m_{\text{secant}} = \text{average velocity} = \frac{\Delta s}{\Delta t} = \frac{s(3.01) - s(3)}{3.01 - 3} = \frac{73.59751 - 73.5}{0.01} = 9.751 \text{ m/s}$$

Approximating the Slope of the Tangent

If we draw a secant for the time interval from $t = 3$ to $t = 3.1$ seconds, the slope of the secant will more closely resemble the slope of the tangent at $t = 3$ seconds.



And lastly, let's try our calculation with an even smaller time interval from 3 to 3.001 seconds.
This gives $\Delta t = 3.001 - 3 = 0.001$.
Since $s(3.001) = -4.9(3.01)^2 + 39.2(3.01) = 73.5097951$ m,

$$m_{\text{secant}} = \text{average velocity} = \frac{\Delta s}{\Delta t} = \frac{s(3.01) - s(3)}{3.01 - 3} = \frac{73.5097951 - 73.5}{0.01} = 9.7951 \text{ m/s}$$

Summarizing Our Results

The table shows the results of our previous calculations and one other calculation that has been done in the same way as the previous results.

Interval	Δt	Average Velocity
$t = 3$ to $t = 3.5$ seconds	0.5 s	7.35 m/s
$t = 3$ to $t = 3.1$ seconds	0.1 s	9.31 m/s
$t = 3$ to $t = 3.01$ seconds	0.01 s	9.75 m/s
$t = 3$ to $t = 3.001$ seconds	0.001 s	9.7951 m/s
$t = 3$ to $t = 3.0001$ seconds	0.0001 s	9.79951 m/s

As Δt becomes smaller and smaller, the average velocity appears to be approaching a certain value.
Graphically, each successive secant is getting closer and closer to the tangent at $t = 3$ seconds.

More specifically, as Δt becomes smaller and smaller,

- the secant more closely resembles the tangent to the curve at the point $(3, 73.5)$,
- the average rate of change becomes a better approximation for the instantaneous rate of change at $t = 3$ seconds, and
- the average velocity becomes a better approximation for the instantaneous velocity at $t = 3$ seconds.

One final note is appropriate here. We could also approach $t = 3$ from below.
That is, we could look at time intervals from $t = 2.9$ to $t = 3$ seconds, $t = 2.99$ to $t = 3$ seconds, and $t = 2.999$ to $t = 3$ seconds.
We would obtain similar results.

Generalizing Our Results

It is reasonable to *approximate* the instantaneous rate of change at $x = a$ by adding a small increment, such as 0.001, to the independent variable.
That is, using $\Delta x = 0.001$.

Then, the slope of the secant between the points $(a, f(a))$ and $(a + 0.001, f(a + 0.001))$ can be found using our slope calculation.

$$\begin{aligned}\text{Instantaneous Rate of Change} = \frac{\Delta f}{\Delta x} &= \frac{f(a+0.001) - f(a)}{a + 0.001 - a} \\ &= \frac{f(a+0.001) - f(a)}{0.001}\end{aligned}$$

The slope of the secant between the points $(a, f(a))$ and $(a + 0.001, f(a + 0.001))$ is a reasonable approximation of the slope of the tangent at $(a, f(a))$.

Examples

Example 2

A kettle is used to heat water. The equation $T(t) = \dfrac{110t + 800}{t + 40}, 0 \leq t \leq 320$, expresses the temperature, T, in degrees Celsius, as a function of time, t, in seconds.

a. Calculate the average rate of change in temperature from $t = 40$ to $t = 50$ seconds.

Solution

Calculating the temperature at $t = 40$ and $t = 50$,

$$T(40) = \frac{110(4) + 800}{40 + 40} = 65°C$$

and

$$T(50) = \frac{110(50) + 800}{50 + 40} = 70°C$$

Then, m_{secant} = average rate of change $= \dfrac{\Delta T}{\Delta t} = \dfrac{T(50) - T(40)}{50 - 40} = 0.5°C/s$.

Examples

Example 2

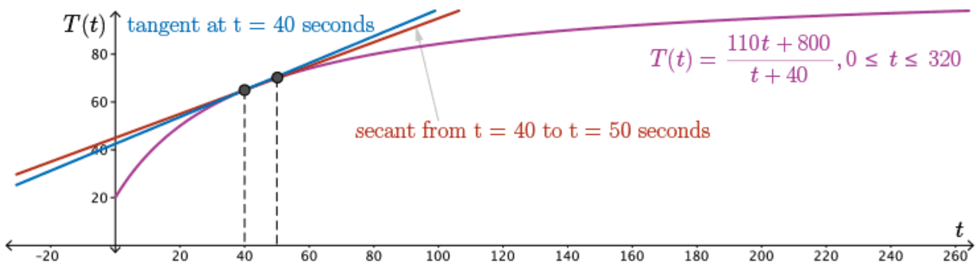
A kettle is used to heat water. The equation $T(t) = \dfrac{110t + 800}{t + 40}, 0 \leq t \leq 320$, expresses the temperature, T, in degrees Celsius, as a function of time, t, in seconds.

b. Approximate the instantaneous rate of change in temperature at 40 seconds, rounded to three decimal places.

Solution

Using an increment of $\Delta t = 0.001$, $T(40.001) = \dfrac{110(40.001) + 800}{40.001 + 40} \approx 65.0006°C$.

The approximated instantaneous rate of change $= \dfrac{\Delta T}{\Delta t} = \dfrac{T(40.001) - T(40)}{40.001 - 40} = 0.562°C/s$.



The slope of the secant is $0.5°C/s$ and the approximate slope of the tangent is $0.562°C/s$.

Examples

Example 2

A kettle is used to heat water. The equation $T(t) = \dfrac{110t + 800}{t + 40}, 0 \leq t \leq 320$, expresses the temperature, T, in degrees Celsius, as a function of time, t, in seconds.

c. What is the significance of the difference between the average rate of change in part a) and the approximation of the instantaneous rate of change at $t = 40$ seconds in part b)?

Solution

As Δt becomes smaller, the slope of the secant (or average rate of change) more closely resembles the slope of the tangent (or instantaneous rate of change).

Summary

In this module, we have

- looked at secants and tangents to curves,
- discussed average rates of change and instantaneous rates of change,
- calculated the average rate of change,
- approximated instantaneous rate of change, and
- applied rates of change to several applications.