Rational Functions Of The Form $y = \dfrac{ax + b}{cx + d}$

In This Module

- We will analyze and graph rational functions of the form

$$y = \frac{ax+b}{cx+d},\ x \neq -\frac{d}{c},\ c \neq 0$$

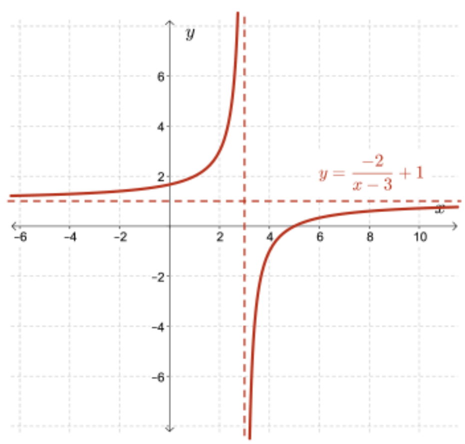
- We will consider the relationship between rational functions of this form and those studied previously, involving transformations of the reciprocal function $y = \dfrac{1}{x}$.

$$y = \frac{ax+b}{cx+d} \qquad \leftrightarrow \qquad y = \frac{a}{b(x-h)} + k$$

Rational Functions in the Form of $y = \frac{ax+b}{cx+d}$

We consider the function $y = \dfrac{-2}{x-3} + 1$. The graph of this function can be obtained by applying transformations to the graph of $y = \dfrac{1}{x}$, in the following order:

- A reflection in the x-axis.
- A vertical stretch from the x-axis by a factor of 2.
- A horizontal translation to the right 3 units.
- A vertical translation up 1 unit.



The domain of the function is $\{x \mid x \neq 3,\ x \in \mathbb{R}\}$.
The graph of this function has a vertical asymptote of $x = 3$ and a horizontal asymptote of $y = 1$.
The range is $\{y \mid y \neq 1,\ y \in \mathbb{R}\}$.
The function is increasing for all values of x in the domain.
There is a zero at $x = 5$.

Rational Functions in the Form of $y = \frac{ax+b}{cx+d}$

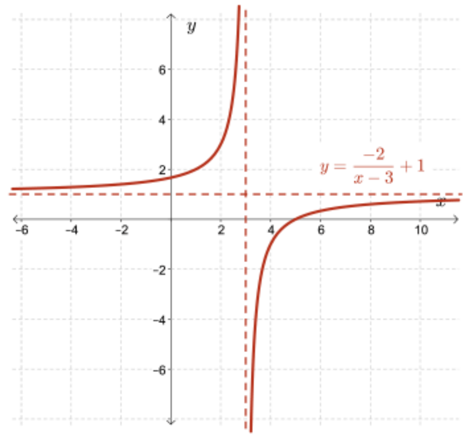
The equation $y = \dfrac{-2}{x-3} + 1$ can be rewritten as

$$y = \frac{-2}{x-3} + \frac{x-3}{x-3}$$

$$y = \frac{x-5}{x-3}$$

which is a rational function of the form $y = \dfrac{ax+b}{cx+d}$.

Do all rational functions of the form $y = \dfrac{ax+b}{cx+d}$ have characteristics similar to reciprocal functions of the form $y = \dfrac{a}{b(x-h)} + k$?



Introduction

$$y = \frac{ax+b}{cx+d} \quad \leftrightarrow \quad y = \frac{a}{b(x-h)} + k$$

Examples

Example 1

Analyze and graph the function $f(x) = \dfrac{3x+6}{x-2}$. State the increasing and/or decreasing intervals, and positive and/or negative intervals of the function.

Solution

We will analyze the behaviour of the function by

- identifying the domain of the function;
- locating the x- and y- intercepts;
- determining the asymptote(s) and/or point(s) of discontinuity, and investigating the behaviour of the graph of the function near them; and
- finding additional points, if necessary.

Examples

Example 1

Analyze and graph the function $f(x) = \dfrac{3x+6}{x-2}$. State the increasing and/or decreasing intervals, and positive and/or negative intervals of the function.

Solution

Domain

The domain of the function is $\{x \mid x \neq 2, x \in \mathbb{R}\}$.

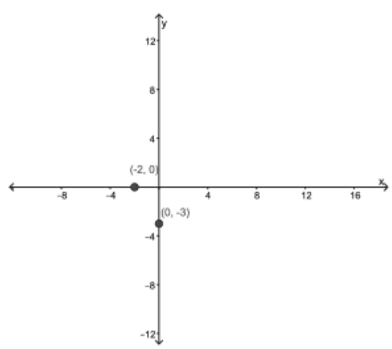
Intercepts

To find the x-intercept(s), set $y = 0$.

$$\begin{aligned} \frac{3x+6}{x-2} &= 0 \\ \therefore 3x+6 &= 0 \\ x &= -2 \end{aligned}$$

To find the y-intercept, set $x = 0$. This gives $y = \dfrac{3(0)+6}{0-2} = -3$.

Therefore, the x-intercept is at $(-2, 0)$ and the y-intercept is at $(0, -3)$.



Examples

Example 1

Analyze and graph the function $f(x) = \dfrac{3x+6}{x-2}$. State the increasing and/or decreasing intervals, and positive and/or negative intervals of the function.

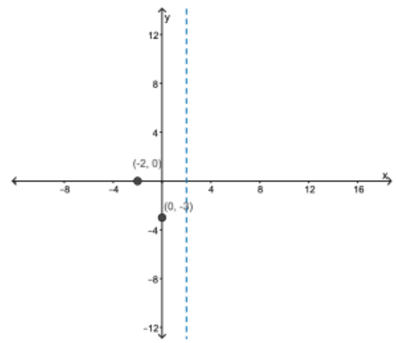
Solution

Vertical Asymptotes/Points of Discontinuity

Since $x \neq 2$ and $f(2) = \dfrac{12}{0}$ (an undefined value), the function has a vertical asymptote of $x = 2$.

To establish the behaviour of the function near the vertical asymptote, $x = 2$, we must determine if $f(x)$ approaches $-\infty$ or $+\infty$ as x approaches 2 from the left and right-hand sides.
That is, we must find

$$\lim_{x \to 2^-} \frac{3x+6}{x-2} \text{ and } \lim_{x \to 2^+} \frac{3x+6}{x-2}$$



Examples

Example 1

Analyze and graph the function $f(x) = \dfrac{3x+6}{x-2}$. State the increasing and/or decreasing intervals, and positive and/or negative intervals of the function.

Solution

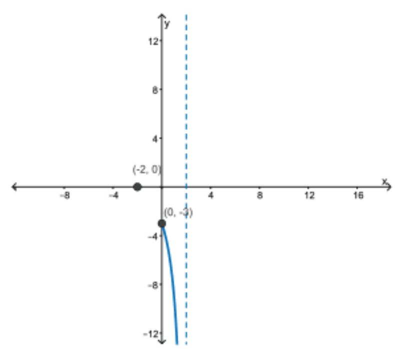
Vertical Asymptotes/Points of Discontinuity

Test: For $x \to 2^-$, substitute $x = 1.999$.

$$y = \frac{3(1.999)+6}{1.999-2} = -11\,997$$

$$\frac{\text{a positive value}}{\text{a very small negative value}} \to \text{a very large negative value}$$

$$\therefore \lim_{x \to 2^-} \frac{3x+6}{x-2} = -\infty$$



Examples

Example 1

Analyze and graph the function $f(x) = \dfrac{3x+6}{x-2}$. State the increasing and/or decreasing intervals, and positive and/or negative intervals of the function.

Solution

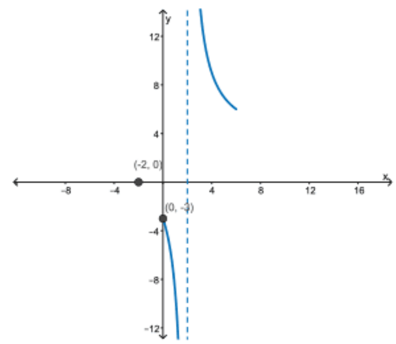
Vertical Asymptotes/Points of Discontinuity

Test: For $x \to 2^+$, substitute $x = 2.001$.

$$y = \frac{3(2.001)+6}{2.001-2} = 12\,003$$

$$\frac{\text{a positive value}}{\text{a very small positive value}} \to \text{a very large positive value}$$

$$\therefore \lim_{x\to 2^+} \frac{3x+6}{x-2} = \infty$$



Examples

Example 1

Analyze and graph the function $f(x) = \dfrac{3x+6}{x-2}$. State the increasing and/or decreasing intervals, and positive and/or negative intervals of the function.

Solution

Horizontal Asymptote

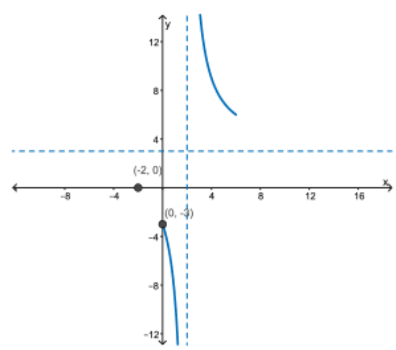
To determine the horizontal asymptote, we must look at the behaviour of the function as $x \to \pm\infty$,

$$\lim_{x\to\infty} \frac{3x+6}{x-2} \text{ and } \lim_{x\to-\infty} \frac{3x+6}{x-2}$$

As $x \to \pm\infty$, the $+6$ in the numerator and the -2 in the denominator become insignificant and $f(x) \to \dfrac{3x}{x}$ so $f(x) \to 3$. Therefore,

$$\lim_{x\to\pm\infty} \frac{3x+6}{x-2} = 3$$

giving $y = 3$ as the horizontal asymptote.



Examples

Example 1

Analyze and graph the function $f(x) = \dfrac{3x+6}{x-2}$. State the increasing and/or decreasing intervals, and positive and/or negative intervals of the function.

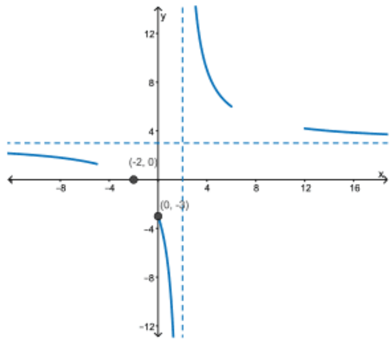
Solution

Horizontal Asymptote

By substituting a large negative and large positive value into the function, we can determine if the graph of the function approaches the horizontal asymptote from above or below.

Since $f(1000) = \dfrac{3(1000)+6}{1000-2} \approx 3.012 > 3,$ the graph approaches $y = 3$ from above as $x \to \infty$.

Since $f(-1000) = \dfrac{3(-1000)+6}{-1000-2} \approx 2.988 < 3,$ the graph approaches $y = 3$ from below as $x \to -\infty$.



Examples

Example 1

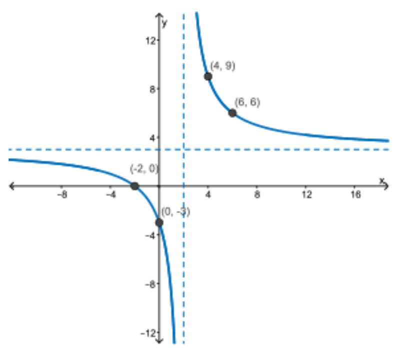
Analyze and graph the function $f(x) = \dfrac{3x+6}{x-2}$. State the increasing and/or decreasing intervals, and positive and/or negative intervals of the function.

Solution

Additional Points

We have two points, $(-2, 0)$ and $(0, -3)$, on the branch of the curve to the left of the vertical asymptote.
We need to identify at least one to the right of the vertical asymptote.
We can use $(4, 9)$ and/or $(6, 6)$ to help complete the graph.
More points may be needed if more accuracy is required.

The range of the function is thus given by $\{y \mid y \neq 3, y \in \mathbb{R}\}$.



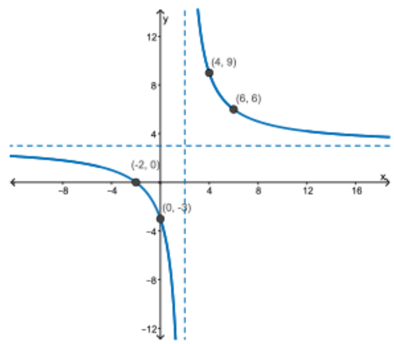
Examples

Example 1

Analyze and graph the function $f(x) = \dfrac{3x+6}{x-2}$. State the increasing and/or decreasing intervals, and positive and/or negative intervals of the function.

Solution

The function is positive when $x \in (-\infty, -2) \cup (2, \infty)$.
The function is negative when $x \in (-2, 2)$.
The curve is decreasing everywhere on its domain.
Therefore, the function is decreasing when $x \in (-\infty, 2) \cup (2, \infty)$, or, more simply, $x \in \mathbb{R},\ x \neq 2$.



Examples

Example 1: Making Connections

We can also use transformations to graph the function $f(x) = \dfrac{3x+6}{x-2}$ by expressing the equation in the form $y = \dfrac{a}{b(x-h)} + k$.

Converting from $y = \dfrac{ax+b}{cx+d}$ **to** $y = \dfrac{a}{b(x-h)} + k$

Method 1

We carry out long division.

$$\begin{array}{r|rr} & & 3 \\ \cline{2-3} x-2 & 3x + & 6 \\ & 3x - & 6 \\ \cline{2-3} & & 12 \end{array}$$

Using the division statement

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

we get

$$f(x) = 3 + \frac{12}{x-2} \quad \text{or} \quad y = \frac{12}{x-2} + 3,\ x \neq 2$$

Examples

Example 1: Making Connections

We can also use transformations to graph the function $f(x) = \dfrac{3x+6}{x-2}$ by expressing the equation in the form $y = \dfrac{a}{b(x-h)} + k$.

Converting from $y = \dfrac{ax+b}{cx+d}$ **to** $y = \dfrac{a}{b(x-h)} + k$

Method 2

We can manipulate the terms of the numerator to allow for the elimination of the x term in the numerator.

$$\begin{aligned} f(x) &= \frac{3x+6}{x-2} \\ &= \frac{3x-6+6+6}{x-2} \\ &= \frac{3(x-2)+12}{x-2} \\ &= \frac{3(x-2)}{x-2} + \frac{12}{x-2} \\ f(x) &= 3 + \frac{12}{x-2},\ x \neq 2 \end{aligned}$$

Examples

Example 1: Making Connections

We can also use transformations to graph the function $f(x) = \dfrac{3x+6}{x-2}$ by expressing the equation in the form $y = \dfrac{a}{b(x-h)} + k$.

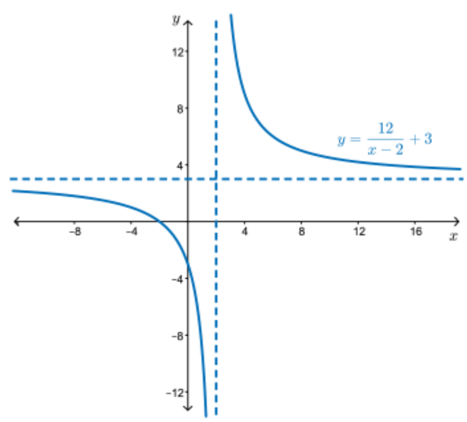
Converting from $y = \dfrac{ax+b}{cx+d}$ **to** $y = \dfrac{a}{b(x-h)} + k$

To obtain the graph of this function,

$$f(x) = \frac{12}{x-2} + 3$$

from the base graph, $y = \dfrac{1}{x}$, apply the following transformations in order:

- A vertical stretch from the x-axis by a factor of 12.
- A horizontal translation to the right 2 units.
- A vertical translation up 3 units.



Examples

Example 2

Graph the function $g(x) = \dfrac{-2x+6}{x-3}$.

Solution

Domain

The domain of the function is $\{x \mid x \neq 3,\ x \in \mathbb{R}\}$.

Intercepts

To find the x-intercept(s), set $y = 0$.

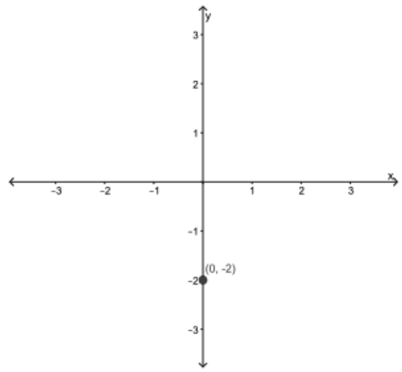
$$\begin{aligned} \frac{-2x+6}{x-3} &= 0 \\ \therefore -2x+6 &= 0 \\ x &= 3 \end{aligned}$$

But $x \neq 3$ (from the domain), so there is no x-intercept.

To find the y-intercept, set $x = 0$. This gives

$g(0) = \dfrac{-2(0)+6}{0-3} = -2$.

Therefore, the y-intercept is at $(0, -2)$.



Examples

Example 2

Graph the function $g(x) = \dfrac{-2x+6}{x-3}$.

Solution

Vertical Asymptotes/Points of Discontinuity

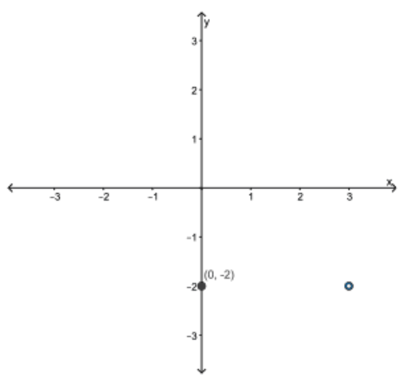
Since $x \neq 3$ and $f(3) = \dfrac{0}{0}$ (an indeterminate value), the function has a point of discontinuity at $x = 3$.

We can support this by considering the behaviour of the function to the left and right side of 3.

$$g(2.999) = -2 \text{ so } \lim_{x \to 3^-} \frac{-2x+6}{x-3} = -2$$

$$g(3.001) = -2 \text{ so } \lim_{x \to 3^+} \frac{-2x+6}{x-3} = -2$$

So $y \to -2$ as $x \to 3$; that is, $\lim_{x \to 3} \dfrac{-2x+6}{x-3} = -2$.



Examples

Example 2

Graph the function $g(x) = \dfrac{-2x+6}{x-3}$.

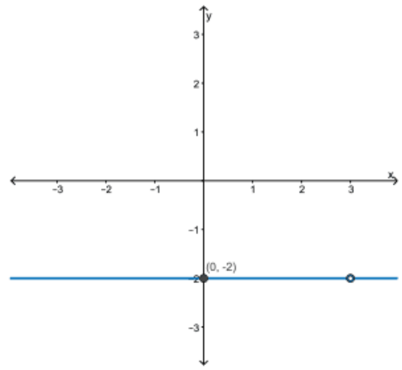
Solution

Vertical Asymptotes/Points of Discontinuity

The equation of the function can be simplified to

$$g(x) = \frac{-2x+6}{x-3},\ x \neq 3$$
$$g(x) = \frac{-2(x-3)}{x-3}$$
$$g(x) = -2,\ x \neq 3$$

Therefore, the graph of the function is the line $y = -2$ with a hole at $(3, -2)$.



Examples

Example 3

Determine an equation for a rational function of the form $f(x) = \dfrac{ax+b}{cx+d}$, $a \neq 0,\ c \neq 0$ that satisfies the following conditions:

$$\lim_{x \to -3^-} f(x) = \infty \qquad \lim_{x \to -3^+} f(x) = -\infty \qquad \lim_{x \to \pm\infty} f(x) = \frac{1}{2}$$

and the function is positive on the interval $x \in (-\infty, -3) \cup (5, \infty)$ and negative on the interval $x \in (-3, 5)$. Graph the function.

Solution

Since $\lim_{x \to -3^-} f(x) = \infty$ and $\lim_{x \to -3^+} f(x) = -\infty$, there is a vertical asymptote at $x = -3$.

Hence, $x+3$ is a factor of the denominator but not the numerator.

$$y = \frac{ax+b}{c(x+3)}$$

Since $\lim_{x \to \pm\infty} f(x) = \dfrac{1}{2}$, the horizontal asymptote is $y = \dfrac{1}{2}$.

Note that $\lim_{x \to \pm\infty} \dfrac{ax+b}{cx+d} = \dfrac{a}{c}$ since b and d become insignificant when $x \to \pm\infty$.

Therefore, the ratio $\dfrac{a}{c} = \dfrac{1}{2}$.

One possible function can be formed with $a = 1$ and $c = 2$: $y = \dfrac{x+b}{2(x+3)}$

Examples

Example 3

Determine an equation for a rational function of the form $f(x) = \dfrac{ax+b}{cx+d}$, $a \neq 0,\ c \neq 0$ that satisfies the following conditions:

$$\lim_{x\to -3^-} f(x) = \infty \qquad \lim_{x\to -3^+} f(x) = -\infty \qquad \lim_{x\to \pm\infty} f(x) = \frac{1}{2}$$

and the function is positive on the interval $x \in (-\infty, -3) \cup (5, \infty)$ and negative on the interval $x \in (-3, 5)$. Graph the function.

Solution

The function changes from positive to negative at $x = -3$ and negative to positive at $x = 5$.
There is a vertical asymptote at $x = -3$, but the graph is continuous at $x = 5$.
Therefore, $x = 5$ is an x-intercept of the function and $y = 0$ when $x = 5$.

$$\begin{aligned} 0 &= \frac{5+b}{2(5+3)} \\ 5+b &= 0 \\ b &= -5 \end{aligned}$$

Therefore, a possible equation for the function is $y = \dfrac{x-5}{2(x+3)}$ or $y = \dfrac{x-5}{2x+6}$.

Examples

Example 3

Determine an equation for a rational function of the form $f(x) = \dfrac{ax+b}{cx+d}$, $a \neq 0,\ c \neq 0$ that satisfies the following conditions:

$$\lim_{x\to -3^-} f(x) = \infty \qquad \lim_{x\to -3^+} f(x) = -\infty \qquad \lim_{x\to \pm\infty} f(x) = \frac{1}{2}$$

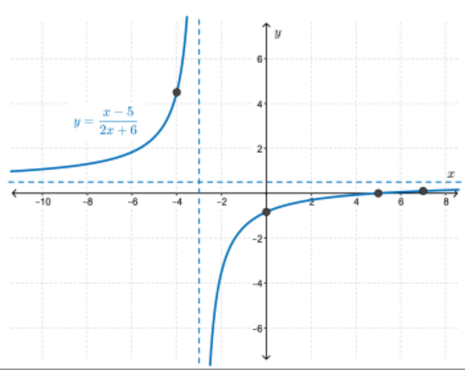
and the function is positive on the interval $x \in (-\infty, -3) \cup (5, \infty)$ and negative on the interval $x \in (-3, 5)$. Graph the function.

Solution

$$y = \frac{x-5}{2x+6}$$

The table shown confirms that the function has the appropriate positive and negative intervals.

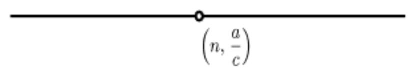
x	y	Function is...
-4	4.5	positive
0	-0.83	negative
7	0.1	positive



Summary

$$y = \frac{ax+b}{cx+d}$$

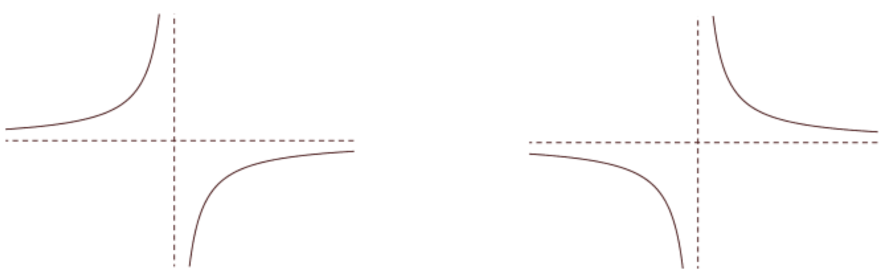
- For all rational functions of the form $y = \dfrac{ax+b}{cx+d}$, $a, c \neq 0$, the domain is $\left\{x \mid x \neq -\dfrac{d}{c},\ x \in \mathbb{R}\right\}$ and the graph of the function is discontinuous at $x = -\dfrac{d}{c}$.
- If the numerator and denominator both equal zero for $x = -\dfrac{d}{c}$ (that is the equation is of the form $y = \dfrac{a(x-n)}{c(x-n)}$, where $n = -\dfrac{d}{c}$, when factored), then the graph is a line defined by $y = \dfrac{a}{c}$, $x \neq n$. Since $x \neq n$, there is a hole at $\left(n, \dfrac{a}{c}\right)$.



Summary

$$y = \frac{ax+b}{cx+d}$$

- Otherwise, the graph has two branches with a vertical and horizontal asymptote and a shape similar to $y = \dfrac{1}{x}$. The branches are symmetrical about the point of intersection of the asymptotes and are either both decreasing or both increasing.



Summary

$$y = \frac{ax+b}{cx+d}$$

- The equation of the vertical asymptote is $x = -\frac{d}{c}$.
- The equation of the horizontal asymptote is $y = \frac{a}{c}$ and the range is $\left\{y \mid y \neq \frac{a}{c}, y \in \mathbb{R}\right\}$.
- The x-intercept is given by $-\frac{b}{a}$.
- The y-intercept is given by $\frac{b}{d}$.