Applications with the Sine and Cosine Functions

Introduction

Through earlier studies in mathematics, a variety of functions have been used to model specific applications. The relationship between a worker's pay and the number of hours worked could be modelled with a simple linear function.

![Graph of Amount Earned Versus Hours Worked]

When a ball is thrown into the air, its height above the ground over time can be modelled using a quadratic function.

![Graph of Height of a Ball Over Time]

Growth and decay situations can be modelled with exponential functions.

In this module, sine and cosine functions will be used to model various physical situations that are periodic.

Examples

Example 1

The pendulum of an antique clock makes 30 complete swings in one minute. A complete swing moves the pendulum from the extreme right point, \( R \), to the extreme left point, \( L \), and back to \( R \) again. The horizontal distance between the two extreme points \( R \) and \( L \) is 36 cm.

a. Determine the period and amplitude of this motion.

Solution

Since there are 30 complete swings in 60 seconds (1 minute),
1 period is \( 60 \div 30 = 2 \) seconds.

The horizontal distance between the two extreme points is 36 cm.
That is, the maximum minus the minimum is 36.
The amplitude is half of this value so the amplitude is 18.
Examples

Example 1

The pendulum of an antique clock makes 30 complete swings in one minute. A complete swing moves the pendulum from the extreme right point, $R$, to the extreme left point, $L$, and back to $R$ again. The horizontal distance between the two extreme points $R$ and $L$ is 36 cm.

b. Draw a sketch to model the pendulum swing for the first 6 seconds. Assume the pendulum starts at $R$.

Solution

Let $d$ represent the horizontal distance from $M$, with positive distance to the right and negative distance to the left.

When the pendulum is at $R$, $d = 18$. When the pendulum is at $L$, $d = -18$.

Since the period is 2 seconds, the pendulum will be at $R$, a maximum, at 0, 2, 4, and 6 seconds.

The minimum will occur at the halfway point of each period so the pendulum will be at $L$ at 1, 3, and 5 seconds.

And halfway between the maximum and minimum, the pendulum will be at $M$.

The pendulum will be at $M$ at 0.5, 1.5, 2.5, 3.5, 4.5, and 5.5 seconds.

A sketch is shown.

Examples

Example 1

The pendulum of an antique clock makes 30 complete swings in one minute. A complete swing moves the pendulum from the extreme right point, $R$, to the extreme left point, $L$, and back to $R$ again. The horizontal distance between the two extreme points $R$ and $L$ is 36 cm.

c. Determine an equation to model the path of the pendulum.

Solution

By referring to the sketch, it would appear that the path of the pendulum can be modelled using a cosine function,

$$d = a \cos[b(x - h)] + k$$

Since a maximum point on the sketch is on the $y$-axis (or in this case, the $d$-axis), there is no horizontal translation and so $h = 0$.

The central horizontal axis is on the $x$-axis (or in this case, the $d$-axis) so there is no vertical translation and $k = 0$.

The amplitude is 18 cm. Since a maximum point is on the positive $y$-axis, there has been no reflection about the $x$-axis and $a = 18$. 
Examples

Example 1

The pendulum of an antique clock makes 30 complete swings in one minute. A complete swing moves the pendulum from the extreme right point, \( R \), to the extreme left point, \( L \), and back to \( R \) again. The horizontal distance between the two extreme points \( R \) and \( L \) is 36 cm.

- Determine an equation to model the path of the pendulum.

Solution

The period is 2 seconds. Thus, \( b = \frac{2\pi}{\text{period}} = \frac{2\pi}{2} = \pi \).

Substituting for \( a = 18 \), \( b = \pi \), \( h = 0 \), and \( k = 0 \) into \( d = a \cos(b(x - h)) + k \), we obtain \( d = 18 \cos(\pi x) \).

If we use \( t \) instead of \( x \), an equation of the function graphed becomes \( d = 18 \cos(\pi t) \), \( 0 \leq t \leq 6 \).

- When \( d < 0 \), the pendulum is closer to \( L \) than it is to \( R \).
- When \( d > 0 \), the pendulum is closer to \( R \) than it is to \( L \).
- When \( d = 0 \), the pendulum is at \( M \).

Examples

Example 1

The pendulum of an antique clock makes 30 complete swings in one minute. A complete swing moves the pendulum from the extreme right point, \( R \), to the extreme left point, \( L \), and back to \( R \) again. The horizontal distance between the two extreme points \( R \) and \( L \) is 36 cm.

- Using the model, determine the exact position of the pendulum at 4.75 seconds.

Solution

Substitute \( t = 4.75 \) into \( d = 18 \cos(\pi t) \):

\[
d = 18 \cos(4.75\pi) \\
= 18 \cos\left(\frac{19\pi}{4}\right) \\
= 18 \cos\left(\frac{3\pi}{4}\right) \\
= 18 \left(-\frac{1}{\sqrt{2}}\right) \\
= 18 \left(-\frac{1}{\sqrt{2}}\right) \times \left(\frac{\sqrt{2}}{\sqrt{2}}\right) \\
= -9\sqrt{2}
\]

At \( t = 4.75 \), the pendulum is located \( 9\sqrt{2} \) cm to the left of the middle position, closer to \( L \) than to \( R \).
Examples

Example 1

The pendulum of an antique clock makes 30 complete swings in one minute. A complete swing moves the pendulum from the extreme right point, $R$, to the extreme left point, $L$, and back to $R$ again. The horizontal distance between the two extreme points $R$ and $L$ is 36 cm.

d. Using the model, determine the exact position of the pendulum at 4.75 seconds.

Solution

At $t = 4.75$, the pendulum is located $9\sqrt{2}$ cm to the left of the middle position, closer to $L$ than to $R$.

We can use the sketch to see that our calculation is reliable:

![Graph showing the position of the pendulum over time.]

Examples

Example 1

The pendulum of an antique clock makes 30 complete swings in one minute. A complete swing moves the pendulum from the extreme right point, $R$, to the extreme left point, $L$, and back to $R$ again. The horizontal distance between the two extreme points $R$ and $L$ is 36 cm.

e. At what times during the first 6 seconds will the pendulum be horizontally 9 cm to the right of its position at $M$?

Solution

Substitute 9 for $d$ into $d = 18 \cos(\pi t)$.

$$9 = 18 \cos(\pi t)$$

$$\frac{1}{2} = \cos(\pi t)$$

Let $\theta = \pi t$. We want $\cos(\theta) = \frac{1}{2}$. The reference angle is $\frac{\pi}{3}$.

Since $\cos(\theta) > 0$, $\theta$ is in quadrant 1 or in quadrant 4.

In quadrant 1, $\theta = \frac{\pi}{3}$ and in quadrant 4, $\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$, for $0 \leq \theta \leq 2\pi$.

Since $\theta = \pi t$,

$$\pi t = \frac{\pi}{3}$$

or

$$\pi t = \frac{5\pi}{3}$$

Dividing by $\pi$,

$$t = \frac{1}{3} \text{ s}$$

and

$$t = \frac{5}{3} = 1 \frac{2}{3} \text{ s}$$

These two possibilities for $t$ lie in the interval $0 \leq t \leq 2$.

To get all times such that $0 \leq t \leq 6$ and multiples of the period length. It follows that the pendulum will be horizontally 9 cm to the right of its position at $M$ for $t = \frac{1}{3}, 1 \frac{2}{3}, 2 \frac{1}{3}, 3 \frac{2}{3}, 4 \frac{1}{3}, 5 \frac{2}{3}$ seconds.
The Ferris Wheel

The Ferris wheel is a ride often found at exhibitions or fairs. Riders enter and exit the ride at the same point.

The circular ride takes its passengers through several revolutions at a constant speed. Our interest in this type of amusement ride is its periodic nature.

We can model this situation with a sinusoidal function which relates the rider’s height above the ground to the length of time on the ride.

Use the worksheet to make connections between various elements of a Ferris wheel ride and features of sinusoidal functions such as period, amplitude, and phase shift.

Examples

Example 2

A Ferris wheel with a radius of 20 m makes one complete revolution every 40 s. The rider’s initial position on the ride is at the bottom of the wheel, 2 m above the ground.

a. Draw a graph showing a rider’s height above the ground during the first two revolutions of the wheel.

Solution

Since one revolution takes 40 s, the period length is 40 s.

The rider gets on at the lowest point so a minimum occurs at (0, 2). Minimums also occur at (40, 2) and (80, 2) since each complete revolution takes 40 s.

The rider gets on at a point 2 m above the ground. The Ferris wheel has a radius of 20 m or a diameter of 40 m. The maximum height above the ground is 2 + 40 or 42 m.

Minimums occur half of a period after each minimum. Therefore, maximums occur at (20, 42) and (60, 42).
**Examples**

**Example 2**

A Ferris wheel with a radius of 20 m makes one complete revolution every 40 s. The rider's initial position on the ride is at the bottom of the wheel, 2 m above the ground.

a. Draw a graph showing a rider's height above the ground during the first two revolutions of the wheel.

**Solution**

![Graph showing a rider's height above the ground](image)

The centre of the wheel is \(2 + 20 = 22\) m above the ground. The rider is at this height at the middle time between adjacent maximum and minimum times.

Therefore, the rider is 22 m above the ground at 10 s, 30 s, 50 s, and 70 s.

A final sketch is shown.

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**Examples**

**Example 2**

A Ferris wheel with a radius of 20 m makes one complete revolution every 40 s. The rider's initial position on the ride is at the bottom of the wheel, 2 m above the ground.

b. Determine an equation modelling the rider's height above the ground during the first two minutes of the ride.

**Solution**

Extend our sketch to 120 seconds (2 minutes).

![Extended graph to 120 seconds](image)

To keep things simpler, we will model with the negative cosine function since the phase shift will be 0. Five points are shown on the graph.

Since the amplitude is 20, let \(a = -20\).

The middle height is 22 m so the central horizontal axis is at \(y = 22\) and it follows that \(k = 22\).

The period is 40 so \(b = \frac{2\pi}{40} = \frac{\pi}{20}\).
Examples

Example 2

A Ferris wheel with a radius of 20 m makes one complete revolution every 40 s. The rider’s initial position on the ride is at the bottom of the wheel, 2 m above the ground.

b. Determine an equation modelling the rider’s height above the ground during the first two minutes of the ride.

Solution

\[
\begin{align*}
\text{Height (m)} & \quad \text{Time (s)} \\
42 & \quad 10 \quad 20 \quad 30 \quad 40 \quad 50 \quad 60 \quad 70 \quad 80 \quad 90 \quad 100 \quad 110 \quad 120 \\
22 & \\
2 & \\
& \\
\end{align*}
\]

Therefore, an equation to model this situation is \( y = -20 \cos\left(\frac{\pi}{20} x\right) + 22 \).

Using variables, which are more representative of the situation,

\[
h = -20 \cos\left(\frac{\pi}{20} t\right) + 22
\]

for \( 0 \leq t \leq 120 \), where \( h \) is the rider’s height, in metres, above the ground at time \( t \) seconds.

Examples

Example 2

A Ferris wheel with a radius of 20 m makes one complete revolution every 40 s. The rider’s initial position on the ride is at the bottom of the wheel, 2 m above the ground.

c. At what times during the first two minutes will the rider’s height above the ground be 38 m or higher?

Solution

\[
\begin{align*}
\text{Height (m)} & \quad \text{Time (s)} \\
42 & \quad 10 \quad 20 \quad 30 \quad 40 \quad 50 \quad 60 \quad 70 \quad 80 \quad 90 \quad 100 \quad 110 \quad 125 \\
38 & \quad \text{draw horizontal line} \\
22 & \\
2 & \\
& \\
\end{align*}
\]

To visualize this, draw the horizontal line \( h = 38 \) on the graph.

We want to determine the times when the sinusoidal function intersects the line and when it is above the line.

Let \( A = \frac{\pi}{20} t \) and \( h = 38 \). Then,

\[
\begin{align*}
38 &= -20 \cos(A) + 22 \\
16 &= -20 \cos(A) \\
-\frac{4}{5} &= \cos(A)
\end{align*}
\]

The reference angle, in radians, is \( \cos^{-1}(0.8) \approx 0.644 \), rounded to three decimal places.
Examples

Example 2

A Ferris wheel with a radius of 20 m makes one complete revolution every 40 s. The rider’s initial position on the ride is at the bottom of the wheel, 2 m above the ground.

c. At what times during the first two minutes will the rider’s height above the ground be 38 m or higher?

Solution

\[ h = -20 \cos \left( \frac{\pi}{20} t \right) + 22 \]

\[ 0 \leq t \leq 120 \] seconds

Since \( \cos A < 0 \), \( A \) is in quadrant 2 or quadrant 3.

In quadrant 2, \( A = \pi - \cos^{-1} \left( \frac{2}{5} \right) \approx 2.498 \) and in quadrant 3, \( A = \pi + \cos^{-1} \left( \frac{2}{5} \right) \approx 3.785 \).

But \( A = \frac{\pi}{20} t \), thus

\[ \frac{\pi}{20} t = 2.498 \]
\[ t = \frac{2.498 \times 20}{\pi} \]
\[ t = 15.9 \text{ s} \]

\[ \frac{\pi}{20} t = 3.785 \]
\[ t = \frac{3.785 \times 20}{\pi} \]
\[ t = 24.1 \text{ s} \]

Examples

Example 2

A Ferris wheel with a radius of 20 m makes one complete revolution every 40 s. The rider’s initial position on the ride is at the bottom of the wheel, 2 m above the ground.

c. At what times during the first two minutes will the rider’s height above the ground be 38 m or higher?

Solution

\[ h = -20 \cos \left( \frac{\pi}{20} t \right) + 22 \]

\[ 0 \leq t \leq 120 \] seconds

For 15.9 \( \leq t \leq 24.1 \), the rider is 38 m or higher above the ground.

Looking back at the graph, this is the first section where the sinusoidal function is on or above the line \( h = 38 \).

To get the other two intervals, add multiples of the period, 40 seconds, to each endpoint of the first interval.

In the first 120 seconds, the rider is 38 m or higher above the ground from 15.9 s to 24.1 s, from 55.9 s to 64.1 s, and from 95.9 s to 104.1 s.
Examples

Example 3
Saint John, New Brunswick, Canada is located on the Bay of Fundy. The depth of the water above sea level is affected by the ocean tides. The following chart gives the depth of the water over a 24-hour period in early December.

<table>
<thead>
<tr>
<th>Hours after Midnight</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water Depth (m)</td>
<td>7.5</td>
<td>7.6</td>
<td>6.9</td>
<td>5.7</td>
<td>4.1</td>
<td>2.5</td>
<td>1.4</td>
<td>1.2</td>
<td>1.9</td>
<td>3.2</td>
<td>4.9</td>
<td>6.6</td>
<td>7.7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Hours after Midnight</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water Depth (m)</td>
<td>8.0</td>
<td>7.4</td>
<td>6.2</td>
<td>4.6</td>
<td>2.9</td>
<td>1.4</td>
<td>0.8</td>
<td>1.1</td>
<td>2.2</td>
<td>3.7</td>
<td>5.4</td>
<td>6.9</td>
</tr>
</tbody>
</table>


a. Draw a graph showing the water depth versus time over the 24-hour period.

Solution

![Graph showing depth of water versus time]

Examples

Example 3
b. Determine the maximum and minimum depths over the 24-hour period. Use these values to estimate the mean depth of the water and then approximate the values for the amplitude and vertical displacement to be used in a sinusoidal model.

Solution

![Graph showing depth of water versus time]

From the data, the maximum depth of the water is 8.0 m occurring 13 hours after midnight. The minimum depth of the water is 0.8 m occurring 19 hours after midnight.

Using these maximum and minimum depths, the mean depth of the water is \( \frac{0.8 + 8.0}{2} = 4.4 \) m.

This information is shown on the graph.

We can now approximate the amplitude and the vertical displacement.

The amplitude is \( a = \frac{8.0 - 0.8}{2} = 3.6 \) and the vertical displacement is \( k = 4.4 \).
Examples

Example 3

c. From the graph, we see that the function appears to be periodic. Estimate the period length.

Solution

There are many ways to arrive at an estimate for the period.

Looking at the horizontal distance between the two maxima and the two minima, the period is close to 12 hours. However, it is known that the period of the tides is between 12 and 13 hours. We will refine our estimate.

Connect the points $A (3, 5.7)$ and $B (4, 4.1)$. This line segment intersects the central horizontal axis at approximately $C (3.81, 4.4)$.

We can determine this intersection by determining the equation of the line passing through $A$ and $B$, then, we find the point where this line crosses the line $y = 4.4$. This will be left as an exercise for you to pursue.

Examples

Example 3

c. From the graph, we see that the function appears to be periodic. Estimate the period length.

Solution

If we connect the points $D (16, 4.6)$ and $E (17, 2.9)$, this line segment intersects the central horizontal axis at approximately $F (16.12, 4.4)$.

An estimate for the period is calculated as the difference between the $x$-coordinate of $F$ and the $x$-coordinate of $C$.

The length of the period is $16.12 - 3.81 = 12.31$ hours or about 12 hours and 19 minutes.
Examples

Example 3

d. Determine an equation to model the depth, \( d \), of the water versus time, \( t \), over the 24-hour period.

Solution

From earlier work with this example, we found that the amplitude was approximately 3.6 and the vertical displacement was 4.4.

Since the period length is 12.31, \( b = \frac{2\pi}{12.31} \approx 0.51 \).

If we use \( C (3.81, 4.4) \) as the first point of a five-point sketch, then the phase shift is 3.81 and so \( h = 3.81 \). We will model with a sine function that has been reflected about the \( t \)-axis but not about the \( d \)-axis.

It follows that \( a = -3.6 \) and \( b = 0.51 \).

Combining all of the information and substituting into \( d = a \sin[b(t - h)] + k \), we obtain

\[
d = -3.6 \sin[0.51(t - 3.81)] + 4.4, \quad 0 \leq t \leq 24
\]

Examples

Example 3

d. Determine an equation to model the depth, \( d \), of the water versus time, \( t \), over the 24-hour period.

Solution

\[
d = -3.6 \sin[0.51(t - 3.81)] + 4.4, \quad 0 \leq t \leq 24
\]

Our model has been added to the plot of the points. We see that our model is reasonably close.

We could also have obtained an equation to model this 24-hour period using the sinusoidal regression function available on most graphing calculators.
Examples

Example 3

e. If a ship requires a minimum of 5 m of water to safely enter the harbour, determine the times in the 24-hour period when it is safe for the ship to enter.

Solution

\[ d = -3.6 \sin(0.51(t - 3.81)) + 4.4 \quad 0 \leq t \leq 24 \]

We want the times when the curve is on or above the line \( d = 5 \). The times can be estimated from the graph or determined algebraically.

Looking at the graph, we estimate that the ship can come into the harbour from midnight to 3:30 AM, from 10:30 AM to 3:30 PM, and from 10:30 PM to midnight.

Examples

Example 3

e. If a ship requires a minimum of 5 m of water to safely enter the harbour, determine the times in the 24-hour period when it is safe for the ship to enter.

Solution

Algebraically, let \( d = 5 \) and let \( \theta = 0.51(t - 3.81) \) in

\[ d = -3.6 \sin(0.51(t - 3.81)) + 4.4 \]

Now, solve the equation

\[ 5 = -3.6 \sin(\theta) + 4.4 \]

Simplifying,

\[ \sin(\theta) = -\frac{0.6}{3.6} = -\frac{1}{6} \]

The reference angle is \( \sin^{-1}\left(\frac{1}{6}\right) \approx 0.1674 \). Since \( \sin(\theta) < 0 \), \( \theta \) is in quadrant 3 or quadrant 4.

In quadrant 3, \( \theta = \pi + \sin^{-1}\left(\frac{1}{6}\right) \approx 3.3090 \) and in quadrant 4, \( \theta = 2\pi - \sin^{-1}\left(\frac{1}{6}\right) \approx 6.1157 \).

But \( \theta = 0.51(t - 3.81) \). Then,

\[ 0.51(t - 3.81) = 3.3090 \quad 0.51(t - 3.81) = 6.1157 \]

\[ t = \frac{3.3090}{0.51} + 3.81 \quad t = \frac{6.1157}{0.51} + 3.81 \]

\[ t \approx 10.30 \text{ h} \quad t \approx 15.80 \text{ h} \]
Examples

Example 3

e. If a ship requires a minimum of 5 m of water to safely enter the harbour, determine the times in the 24-hour period when it is safe for the ship to enter.

Solution

We want all possible values for $t$ such that $0 \leq t \leq 24$ so we need to determine all possible coterminal angles in this interval.

The period is 12.31, so other possible values for $t$ are $10.3 + 12.31 = 22.61$ and $15.80 - 12.31 = 3.49$.

Adding or subtracting other multiples of the period will produce values outside the required domain.

The depth of the water is 5 m when $t = 3.49$ h, $t = 10.30$ h, $t = 15.80$ h, or $t = 22.61$ h.

Converting these to more recognizable times, the depth is 5 m at 3:29 AM, at 10:18 AM, at 3:48 PM, and at 10:37 PM.

Using these times in combination with the graph, the ship can enter the harbour safely from midnight to 3:29 AM, from 10:18 AM to 3:48 PM, and from 10:37 PM to midnight.

Summary

There are many real-world applications which are periodic that can be modelled using the sine or cosine functions. In this module, we have attempted to illustrate a few of these examples and solve related problems.

Motion of a Ferris Wheel

![Diagram of Ferris Wheel]

Changing Depths of Water Due to Tidal Influence

![Graph showing changing depths of water over time]