Graphs Of Reciprocal Trigonometric Functions

Introduction

To sketch the reciprocal trigonometric functions, we could use a table of values approach as we did with primary trigonometric ratios in a previous module.

We also employed a table of values approach when we sketched $y = x^2$, $y = \sqrt{x}$, $y = 2^x$, $y = |x|$, and $y = \frac{1}{x}$, earlier in this course.

However, for the reciprocal function $y = \frac{1}{x}$, we provided a second approach.

We will review this second approach and we will generalize it in this module to sketch the three reciprocal trigonometric functions.

Primary Trigonometric Functions	Reciprocal Trigonometric Functions
$y = \sin(x)$	$y = \csc(x) = \dfrac{1}{\sin(x)}$
$y = \cos(x)$	$y = \sec(x) = \dfrac{1}{\cos(x)}$
$y = \tan(x)$	$y = \cot(x) = \dfrac{1}{\tan(x)}$

Sketching the Reciprocal Function

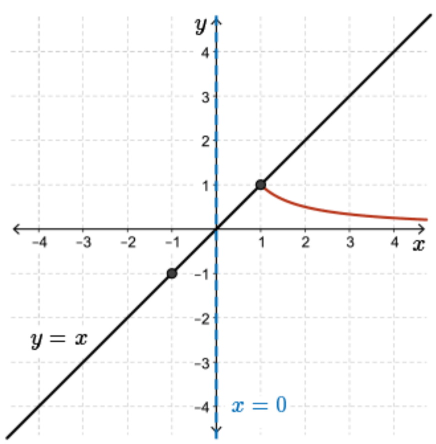
Sketch the graph of $f(x) = \frac{1}{x}$

1. Sketch the line $g(x) = x$. In general, given $f(x) = \dfrac{1}{g(x)}$, we graph the denominator function $y = g(x)$.

2. When $g(x) = \pm 1$, $f(x) = \dfrac{1}{g(x)} = \pm 1$. In this example, this occurs when $x = \pm 1$. The points $(1, 1)$ and $(-1, -1)$ are on both $g(x) = x$ and $f(x) = \frac{1}{x}$. Plot these two points.

3. If $g(x) = 0$, $f(x) = \dfrac{1}{g(x)}$ is undefined. Draw a vertical asymptote along the y-axis.

4. As values of x go from 1 to $+\infty$, $g(x)$, the denominator function, approaches ∞ and so $f(x) = \dfrac{1}{g(x)}$ gets closer and closer to zero, the x-axis, from above.



Sketching the Reciprocal Function

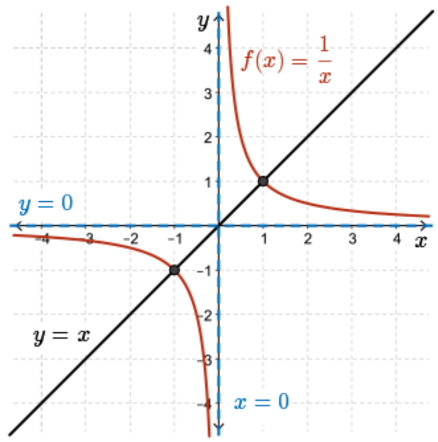
Sketch the graph of $f(x) = \dfrac{1}{x}$

5. As values of x go from -1 to $-\infty$, $g(x)$, the denominator function, approaches $-\infty$ and so $f(x) = \dfrac{1}{g(x)}$ gets closer and closer to zero, the x-axis, from below.

6. Draw in the horizontal asymptote along the x-axis.

7. As values of x start from -1 and approach 0 from the left, $g(x)$, the denominator function, approaches 0 from below and so $f(x) = \dfrac{1}{g(x)}$ approaches $-\infty$.

8. As values of x start from 1 and approach 0 from the right, $g(x)$, the denominator function, approaches 0 from above and so $f(x) = \dfrac{1}{g(x)}$ approaches ∞.



Sketching the Reciprocal Function

General steps for sketching $f(x) = \dfrac{1}{g(x)}$

1. Sketch the function $y = g(x)$.
2. Identify the values of x where $g(x) = 1$ or $g(x) = -1$. At these points $f(x) = g(x)$. That is, these points are on both $f(x)$ and $g(x)$. These points are called **fixed points** or **static points**.
3. Identify the x-intercepts of $g(x)$. At these points, $f(x)$ is undefined. There will be vertical asymptotes for these values of x.
4. If required, determine what happens to the reciprocal function as x approaches the vertical asymptotes from the left and from the right.
5. If required, determine the end behaviour of $f(x)$.

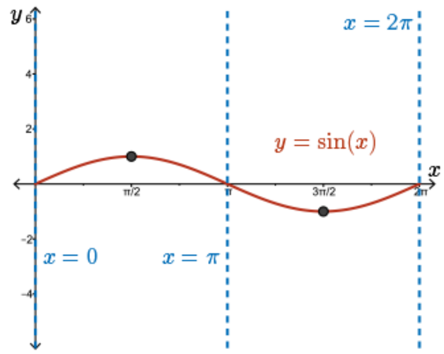
Sketching $y = \csc(x)$ from the graph of $y = \sin(x)$

Sketch the graph of $y = \csc(x)$, $0 \leq x \leq 2\pi$

1. Since $\csc(x) = \dfrac{1}{\sin(x)}$, sketch $y = \sin(x)$, $0 \leq x \leq 2\pi$.

2. Identify the points which are on the graphs of both $y = \sin(x)$ and $y = \csc(x)$. We want the values of x for which $\sin(x) = 1$ or $\sin(x) = -1$. For $0 < x \leq 2\pi$, $\sin(\frac{\pi}{2}) = 1$ and $\sin(\frac{3\pi}{2}) = -1$. The points $\left(\dfrac{\pi}{2}, 1\right)$ and $\left(\dfrac{3\pi}{2}, -1\right)$ are on both functions. Plot these points.

3. Identify the x-intercepts of $y = \sin(x)$. At these values of x, $y = \csc(x)$ is undefined and there will be vertical asymptotes at $x = 0$, $x = \pi$, and $x = 2\pi$.

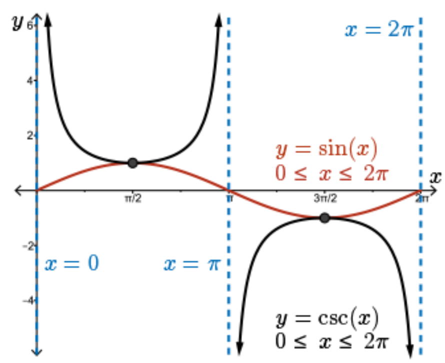


Sketching $y = \csc(x)$ from the graph of $y = \sin(x)$

Sketch the graph of $y = \csc(x)$, $0 \leq x \leq 2\pi$

4. Determine what happens to the reciprocal function between the asymptotes.

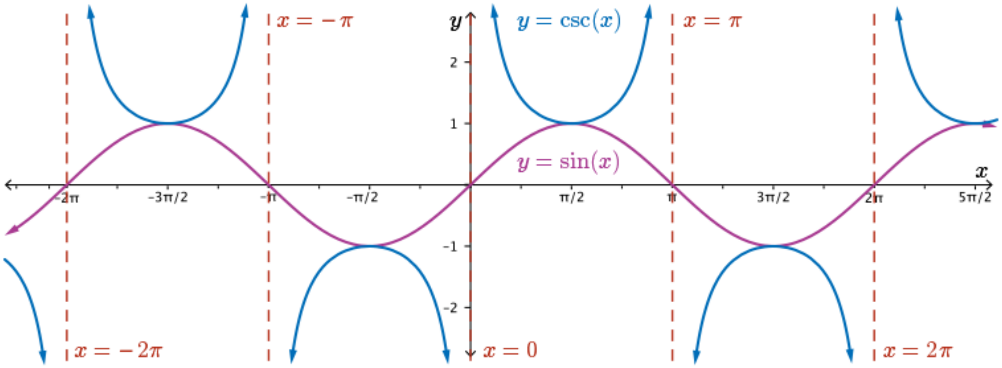
- As the value of x moves from $\dfrac{\pi}{2}$ towards 0, the value of $\sin(x)$ goes from 1 to 0. Then the value of the reciprocal goes from 1 to ∞.
- As the value of x moves from $\dfrac{\pi}{2}$ towards 1, the value of $\sin(x)$ goes from 1 to 0. Then the value of the reciprocal goes from 1 to ∞.
- As the value of x moves from $\dfrac{3\pi}{2}$ towards π, the value of $\sin(x)$ goes from -1 to 0, approaching from below. Then the value of the reciprocal goes from -1 to $-\infty$.
- As the value of x moves from $\dfrac{3\pi}{2}$ towards 2π, the value of $\sin(x)$ goes from -1 to 0, approaching from below. Then the value of the reciprocal goes from -1 to $-\infty$.



Sketching $y = \csc(x)$ from the graph of $y = \sin(x)$

Properties of $y = \csc(x)$

It is straightforward to see that the cosecant function will be periodic and have the same period as the sine function. For the cosecant function, state the x and y-intercepts, the period and the equations of any asymptotes, and the domain and range.



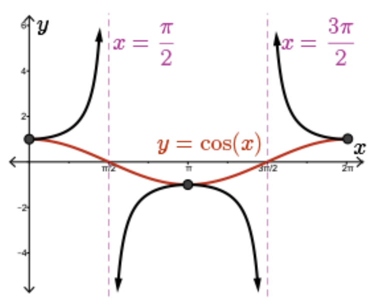
There are no x-intercepts. There is no y-intercept. The period of $y = \csc(x)$ is 2π.
The vertical asymptotes occur every π units with one of them at $x = 0$. Therefore, the equations of the vertical asymptotes are $x = n\pi, n \in \mathbb{Z}$. There are no horizontal asymptotes. The domain is $\{x \mid x \neq n\pi, x \in \mathbb{R}, n \in \mathbb{Z}\}$.
No values of $y = \csc(x)$ lie between -1 and 1. The range of $y = \csc(x)$ is $\{y \mid y \leq -1 \text{ or } y \geq 1, y \in \mathbb{R}\}$.
The range can also be written using absolute value notation as $\{y \mid |y| \geq 1, y \in \mathbb{R}\}$.

Sketching $y = \sec(x)$ from the graph of $y = \cos(x)$

Sketch the graph of $y = \sec(x)$, $0 \geq x \geq 2\pi$ from its reciprocal function $y = \cos(x)$

1. Since $\cos(x) = 1$ when $x = 0$ and $x = 2\pi$, $\sec(x) = 1$ when $x = 0$ and $x = 2\pi$. Since $\cos(x) = -1$ when $x = \pi$, $\sec(x) = -1$ when $x = \pi$. The three points $(0, 1)$, $(\pi, -1)$ and $(2\pi, 1)$ are on both $y = \cos(x)$ and $y = \sec(x)$.

2. The x-intercepts of $y = \cos(x)$ occur when $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. The secant function will therefore have vertical asymptotes at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.



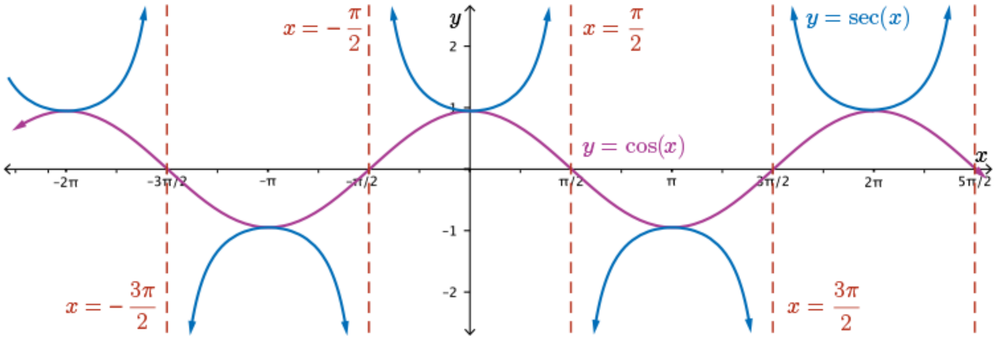
3. Now we look at how the secant function behaves as it approaches each vertical asymptote from either side.

- As x moves towards $\frac{\pi}{2}$ from the left, $\cos(x)$ goes from 1 to 0 and $\sec(x)$ goes from 1 to ∞.
- As x moves towards $\frac{\pi}{2}$ from the right, $\cos(x)$ goes from -1 to 0 and $\sec(x)$ goes from -1 to $-\infty$.
- As x moves towards $\frac{3\pi}{2}$ from the left, $\cos(x)$ goes from -1 to 0 and $\sec(x)$ goes from -1 to $-\infty$.
- As x moves towards $\frac{3\pi}{2}$ from the right, $\cos(x)$ goes from 1 to 0 and $\sec(x)$ goes from 1 to ∞.

Sketching $y = \sec(x)$ from the graph of $y = \cos(x)$

Properties of $y = \sec(x)$

For the secant function, state the x and y-intercepts, the period and the equations of any asymptotes, and the domain and range.



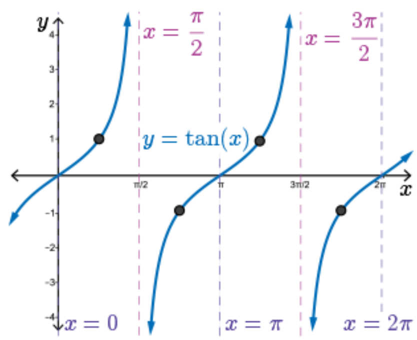
There are no x-intercepts. The y-intercept of the secant function is 1. The period of $y = \sec(x)$ is 2π. Vertical asymptotes occur every π units with one at $x = \frac{\pi}{2}$. Therefore, the equations of the vertical asymptotes are $x = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}$. There are no horizontal asymptotes. The domain is $\left\{x \mid x \neq \frac{\pi}{2} + n\pi, x \in \mathbb{R}, n \in \mathbb{Z}\right\}$. No values of $y = \sec(x)$ lie between -1 and 1. The range of $y = \sec(x)$ is $\{y \mid y \leq -1 \text{ or } y \geq 1, y \in \mathbb{R}\}$. The range can also be written using absolute value notation $\{y \mid |y| \geq 1, y \in \mathbb{R}\}$.

Sketching $y = \cot(x)$ from the graph of $y = \tan(x)$

Sketch the graph of $y = \cot(x)$, $0 \leq x \leq 2\pi$ from its reciprocal function $y = \tan(x)$.

1. Since $\tan(x) = 1$ when $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$, $\cot(x) = 1$ when $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$. Since $\tan(x) = -1$ when $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$, $\cot(x) = -1$ when $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$. The four points $\left(\frac{\pi}{4}, 1\right)$, $\left(\frac{3\pi}{4}, -1\right)$, $\left(\frac{5\pi}{4}, 1\right)$, and $\left(\frac{7\pi}{4}, -1\right)$ are on both $y = \tan(x)$ and $y = \cot(x)$.

2. The x-intercepts of $y = \tan(x)$ occur when $x = 0, x = \pi$, and $x = 2\pi$. The cotangent function will therefore have vertical asymptotes for these values of x.

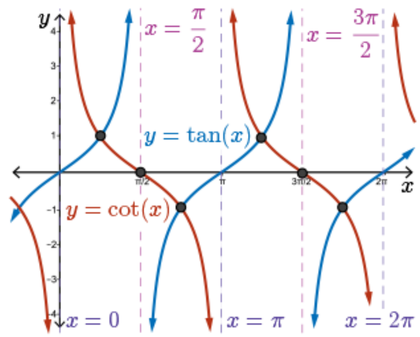


Sketching $y = \cot(x)$ from the graph of $y = \tan(x)$

Sketch the graph of $y = \cot(x)$, $0 \leq x \leq 2\pi$ from its reciprocal function $y = \tan(x)$.

3. The asymptotes of $y = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$ occur when $x = \dfrac{\pi}{2}$ and $x = \dfrac{3\pi}{2}$ since the denominator $\cos(x)$ equals 0 for these values of x. At these same values of x, the numerator $\sin(x)$ is either 1 or -1. In the reciprocal $y = \cot(x) = \dfrac{\cos(x)}{\sin(x)}$, for the same values of x, the numerator $\cos(x)$ equals 0 and the denominator $\sin(x)$ is either 1 or -1. In either case, the value of $y = \cot(x)$ is 0.

Therefore, the x-intercepts of $y = \cot(x)$ occur when $x = \dfrac{\pi}{2}$ and $x = \dfrac{3\pi}{2}$.



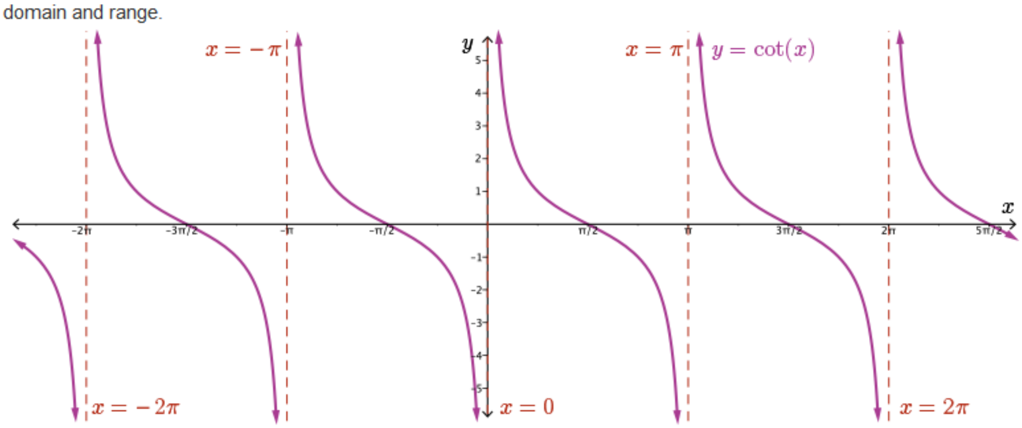
4. The argument for the behaviour around the asymptotes is similar to the argument around the asymptotes for $y = \csc(x)$ and $y = \sec(x)$, and will not be repeated here.

The final graph is shown for you.

Sketching $y = \cot(x)$ from the graph of $y = \tan(x)$

Properties of $y = \cot(x)$

For the cotangent function, state the x and y-intercepts, the period and the equations for any asymptotes, and the domain and range.



The x-intercepts are $x = \pm\frac{3\pi}{2}$, $x = \pm\frac{\pi}{2}$, and $x = \frac{5\pi}{2}$. In general, the x-intercepts are $x = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}$. There is no y-intercept. The period of $y = \cot(x)$ is π. There are no horizontal asymptotes. The vertical asymptotes occur every π units with one of them at $x = 0$. Therefore, the equations of the vertical asymptotes can be written $x = n\pi, n \in \mathbb{Z}$. The domain is $\{x \mid x \neq n\pi, x \in \mathbb{R}, n \in \mathbb{Z}\}$. The range is $\{y \mid y \in \mathbb{R}\}$.

Summary

In this module, we have sketched the three functions:
$y = \csc(x), y\sec(x)$, and $y = \cot(x)$.
We have also looked at properties of each.
In upcoming modules, we will look at transformations of $y = \sin(x)$ and $y = \cos(x)$.