Working with Sinusoidal Functions

Recall

In an earlier module, we looked at the transformational form of two parent sinusoidal functions $y = \sin(x)$ and $y = \cos(x)$.

The Sine Function
$y = a\sin[b(x - h)] + k$

The Cosine Function
$y = a\cos[b(x - h)] + k$

We graphed sinusoidal functions using a variety of techniques.
We discussed the features of the functions and how to determine them from the equation.

- The amplitude is $|a|$ and can be determined by the formula $\dfrac{\text{maximum} - \text{minimum}}{2}$.
  If $a < 0$, the parent function is reflected about the x-axis.
- The period is $\dfrac{2\pi}{|b|}$ radians or $\dfrac{360^\circ}{|b|}$.
  If $b < 0$, the parent function is reflected about the y-axis.
- The phase shift is h.
- The vertical displacement is k and can be calculated using $\dfrac{\text{maximum} + \text{minimum}}{2}$.
  k is the average of the maximum and minimum values of the function.
  The equation of the central horizontal axis is $y = k$.
- The maximum value is $k + |a|$ and the minimum value is $k - |a|$.

Examples

Example 1

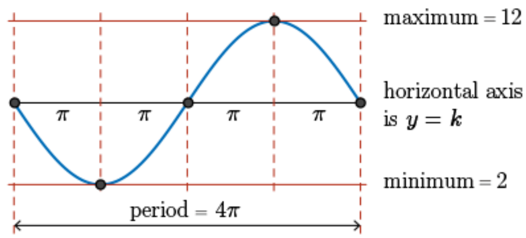
Determine an equation for a sine function that has been reflected about the x-axis, has a maximum value 12, a minimum value 2, a period 4π, and a phase shift $-\dfrac{\pi}{3}$.

Solution

A quick sketch of one period is shown.
The equation is of the form

$$y = a\sin[b(x - h)] + k$$

Since there is a reflection about the x-axis, $a < 0$.
To determine the amplitude, we subtract the minimum value from the maximum value and divide the result by 2.



The amplitude is $\dfrac{12 - 2}{2} = \dfrac{10}{2} = 5$ and it follows that $a = -5$.

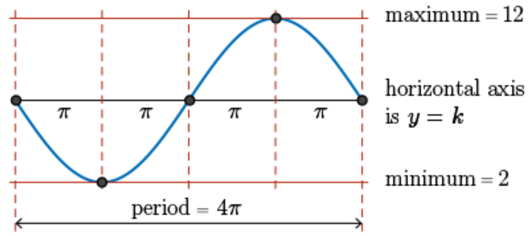
Examples

Example 1

Determine an equation for a sine function that has been reflected about the x-axis, has a maximum value 12, a minimum value 2, a period 4π, and a phase shift $-\frac{\pi}{3}$.

Solution

We know that k is the average of the maximum and minimum values so $k = \frac{12-2}{2} = \frac{14}{2} = 7$. (It follows that the equation of the central horizontal axis is $y = 7$.)



Since the phase shift is given, we know that $h = -\frac{\pi}{3}$.

We know that the period $= \frac{2\pi}{|b|}$.

Rearranging, $|b| = \frac{2\pi}{\text{period}} = \frac{2\pi}{4\pi} = \frac{1}{2}$.

Since there is no reflection about the y-axis, $b > 0$ and it follows that $b = \frac{1}{2}$. Combining the information, we have $a = -5, b = \frac{1}{2}, h = -\frac{\pi}{3}$ and $k = 7$. A possible equation is $y = -5\sin\left[\frac{1}{2}\left(x + \frac{\pi}{3}\right)\right] + 7$.

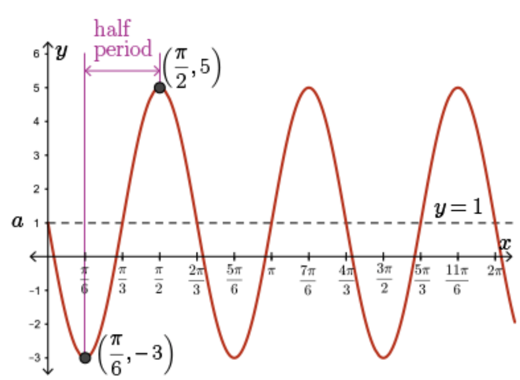
Examples

Example 2

A sinusoidal function is defined for $x \geq 0$. The first minimum occurs as $\left(\frac{\pi}{6}, -3\right)$ and the first maximum occurs at $\left(\frac{\pi}{2}, 5\right)$. Determine a possible equation for this function.

Solution

A sketch showing the given information is not very helpful it would seem.



However, if we draw a sinusoidal curve through the adjacent minimum and maximum points, we may be able to acquire more information.

The maximum value is 5 and the minimum value is -3.

k is the average of these two values so $k = 1$ and the vertical displacement is 1.

The amplitude is $\frac{5-(-3)}{2} = 4$ so $|a| = 4$.

The period can be found by determining the horizontal distance from one minimum point to the next minimum point.

The horizontal distance from one minimum point to the next maximum point is half of a period.

So half of a period is $\frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$.

Examples

Example 2

A sinusoidal function is defined for $x \geq 0$. The first minimum occurs as $\left(\frac{\pi}{6}, -3\right)$ and the first maximum occurs at $\left(\frac{\pi}{2}, 5\right)$. Determine a possible equation for this function.

Solution

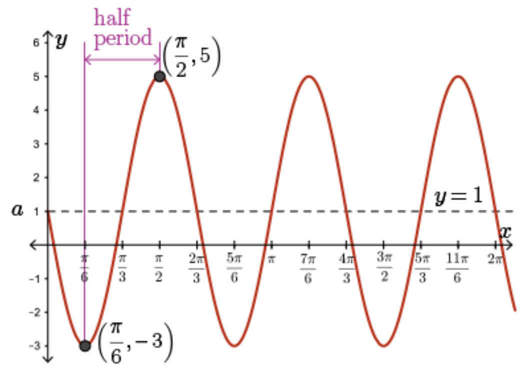
It follows that the full period length is $2 \times \frac{\pi}{3} = \frac{2\pi}{3}$ radians.

At this point we can calculate $|b|$.

Recall that period $= \frac{2\pi}{|b|}$.

After rearranging, $|b| = \frac{2\pi}{\text{period}} = \frac{2\pi}{\frac{2\pi}{3}} = 3$.

We still must decide which sinusoidal function to model with.



Examples

Example 2

A sinusoidal function is defined for $x \geq 0$. The first minimum occurs as $\left(\frac{\pi}{6}, -3\right)$ and the first maximum occurs at $\left(\frac{\pi}{2}, 5\right)$. Determine a possible equation for this function.

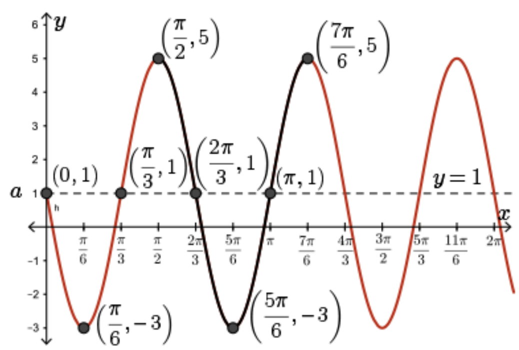
Solution

From our work, we know that $|a| = 4$, $|b| = 3$, the period is $\frac{2\pi}{3}$, and $k = 1$.

Using our knowledge of the behaviour of a sinusoidal function and the fact that one-quarter of a period is $\frac{2\pi}{3} \div 4 = \frac{\pi}{6}$, we can determine the coordinates of more points on the graph.

Several more points have been added to the graph.

If we model with a cosine function, we could use the point $\left(\frac{\pi}{2}, 5\right)$ as the first point in a five-point sketch.



We would use a phase shift of $h = \frac{\pi}{2}$, $a = 4$, $b = 3$, and $k = 1$ resulting in the equation

$$y = 4\cos\left[3\left(x - \frac{\pi}{2}\right)\right] + 1.$$

Examples

Example 2

A sinusoidal function is defined for $x \geq 0$. The first minimum occurs as $\left(\frac{\pi}{6}, -3\right)$ and the first maximum occurs at $\left(\frac{\pi}{2}, 5\right)$. Determine a possible equation for this function.

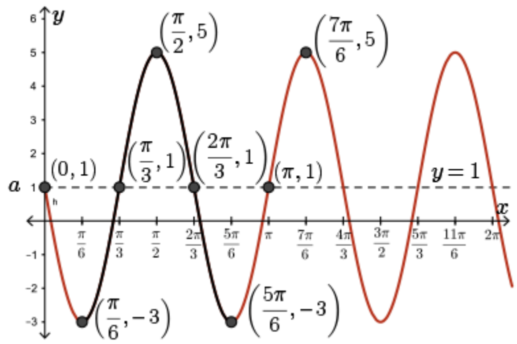
Solution

If we model with a cosine function which has been reflected in the x-axis, we could use the point $\left(\frac{\pi}{6}, -3\right)$ as the first point in a five-point sketch.

We would use a phase shift of $h = \frac{\pi}{6}$, $a = -4, b = 3,$ and $k = 1$, resulting in the equation

$y = -4\cos\left[3\left(x - \frac{\pi}{6}\right)\right] + 1.$

If we model with the sine function, we could use the point $\left(\frac{\pi}{3}, 1\right)$ as the first point in a five-point sketch.



Examples

Example 2

A sinusoidal function is defined for $x \geq 0$. The first minimum occurs as $\left(\frac{\pi}{6}, -3\right)$ and the first maximum occurs at $\left(\frac{\pi}{2}, 5\right)$. Determine a possible equation for this function.

Solution

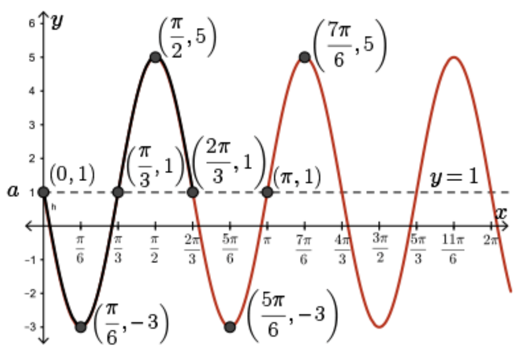
We would use a phase shift of $h = \frac{\pi}{3}$, $a = 4, b = 3$ and $k = 1$, resulting in the equation

$y = 4\sin\left[3\left(x - \frac{\pi}{3}\right)\right] + 1.$

If we model with the sine function which has been reflected in the x-axis, we could also use the point $(0, 1)$ as the first point in a five-point sketch.

We would use a phase shift of $h = 0, a = -4, b = 3,$ and $k = 1$, resulting in the equation

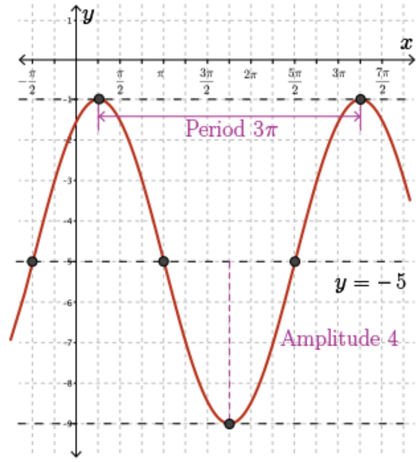
$y = -4\sin(3x) + 1.$



Examples

Example 3

A sinusoidal function is shown on the following graph. Determine a possible equation for this graph.



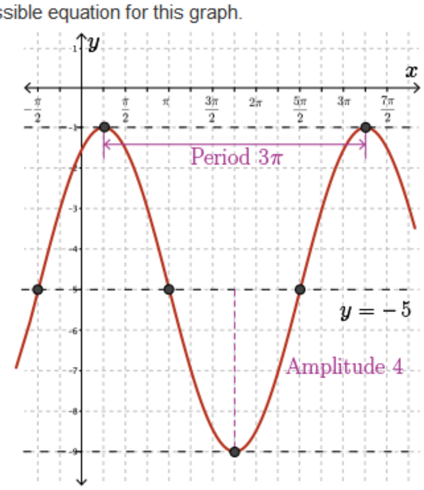
Solution

First, we will determine the maximum and minimum values.
One way to do this is to draw horizontal lines tangent to the curve through maximum and minimum points.
As a result of this, we get several pieces of information.
The maximum is -1, the minimum is -9, and the amplitude is 4.
It follows that $|a| = 4$.
The central horizontal axis is $y = -5$.
From this, we get that the vertical displacement is $k = -5$.
By using the x-coordinates of two consecutive maximum points, we determine that the period is $\frac{13\pi}{4} - \frac{\pi}{4} = 3\pi$.
It follows that $|b| = \frac{2\pi}{3\pi} = \frac{2}{3}$.
We can plot key points on the graph: maximum points, minimum points, and points on the central horizontal axis.

Examples

Example 3

A sinusoidal function is shown on the following graph. Determine a possible equation for this graph.



Solution

We can read the phase shift from the graph and select an appropriate sinusoidal function.

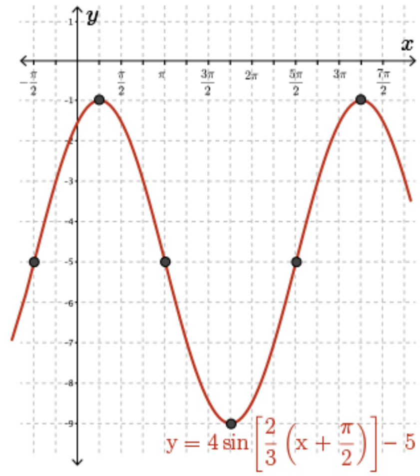
- For a phase shift $h = -\frac{\pi}{2}$, $a = 4, b = \frac{2}{3}$, $k = -5$, and $y = 4\sin\left[\frac{2}{3}\left(x + \frac{\pi}{2}\right)\right] - 5$.
- For a phase shift $h = \frac{\pi}{4}$, $a = 4, b = \frac{2}{3}$, $k = -5$, and $y = 4\cos\left[\frac{2}{3}\left(x - \frac{\pi}{4}\right)\right] - 5$.
- For a phase shift $h = \pi, a = -4, b = \frac{2}{3}$, $k = -5$, and $y = -4\cos\left[\frac{2}{3}(x - \pi)\right] - 5$.
- For a phase shift $h = \frac{7\pi}{4}$, $a = -4, b = \frac{2}{3}$, $k = -5$, and $y = -4\cos\left[\frac{2}{3}\left(x - \frac{7\pi}{4}\right)\right] - 5$.

Examples

Example 4

In the previous example, we developed several equations which could be used to model the given information.
Once we have an equation, we are generally interested in obtaining more information, either from the graph or from the equation.
Using the graph, determine all possible values of x such that the value of the sinusoidal function is a maximum.

Solution



From the graph, we can see that a maximum of -1 occurs when $x = \frac{\pi}{4}$.
From our earlier work, we know that the period is 3π.
Therefore, the maximum value -1 occurs when

$$x = \frac{\pi}{4} + 3\pi n, n \in \mathbb{Z}$$

Examples

Example 5

Using the equation $y = 4\sin\left[\frac{2}{3}\left(x + \frac{\pi}{2}\right)\right] - 5$, determine all possible values of x such that the value of the sinusoidal function is a maximum.

Solution

From the equation, we see that $a = 4$, $b = \frac{2}{3}$, $h = -\frac{\pi}{2}$, and $k = -5$. Using $a = 4$ and $k = -5$, the maximum value is $-5 + 4 = -1$, as we saw from the graph. The period is $\frac{2\pi}{|b|} = \frac{2\pi}{\frac{2}{3}} = 3\pi$.

The angle $\frac{2}{3}\left(x + \frac{\pi}{2}\right)$ looks a bit complex so we can simplify if we let $A = \frac{2}{3}\left(x + \frac{\pi}{2}\right)$.

We know that for $y = \sin(A)$, a maximum occurs when $A = \frac{\pi}{2}$.

$$\frac{2}{3}\left(x + \frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$x + \frac{\pi}{2} = \frac{3\pi}{4} \qquad \text{multiplying both sides of the equation by } \frac{3}{2}$$

$$x = \frac{3\pi}{4} - \frac{\pi}{2}$$

$$x = \frac{\pi}{4}$$

Examples

Example 5

Using the equation $y = 4\sin\left[\frac{2}{3}\left(x + \frac{\pi}{2}\right)\right] - 5$, determine all possible values of x such that the value of the sinusoidal function is a maximum.

Solution

Since we know that the period is 3π radians, the maximum value -1 occurs when $x = \frac{\pi}{4} + 3\pi n, n \in \mathbb{Z}$, as expected.

This same approach will also work to determine when the minimum values or middle values occur.

But what if we want other values? We can approximate from the graph or solve the sinusoidal equation for the required values.

Examples

Example 6

Determine the values of $x, 0 \leq x \leq 3\pi$ so that $y = 4\sin\left[\frac{2}{3}\left(x + \frac{\pi}{2}\right)\right] - 5$ equals -3.

Solution

The angle $\frac{2}{3}\left(x + \frac{\pi}{2}\right)$ looks a bit complex so we can simplify if we let $A = \frac{2}{3}\left(x + \frac{\pi}{2}\right)$.

We will solve an equation which is much more straightforward.

$$4\sin(A) - 5 = -3$$
$$4\sin(A) = 2$$
$$\sin(A) = \frac{1}{2}$$

We know that the reference angle is $\frac{\pi}{6}$ and $\sin(A) > 0$ in quadrants 1 and 2. It follows that $A = \frac{\pi}{6}$ in quadrant 1 and $A = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ in quadrant 2.

Examples

Example 6

Determine the values of $x, 0 \le x \le 3\pi$ so that $y = 4\sin\left[\frac{2}{3}\left(x + \frac{\pi}{2}\right)\right] - 5$ equals -3.

Solution

Since $A = \frac{2}{3}\left(x + \frac{\pi}{2}\right)$, we know that

$$\begin{aligned} \frac{2}{3}\left(x + \frac{\pi}{2}\right) &= \frac{\pi}{6} \\ x + \frac{\pi}{2} &= \frac{\pi}{6} \times \frac{3}{2} \\ x + \frac{\pi}{2} &= \frac{\pi}{4} \\ x &= \frac{\pi}{4} - \frac{\pi}{2} \\ x &= -\frac{\pi}{4} \end{aligned}$$

$$\begin{aligned} \frac{2}{3}\left(x + \frac{\pi}{2}\right) &= \frac{5\pi}{6} \\ x + \frac{\pi}{2} &= \frac{5\pi}{6} \times \frac{3}{2} \\ x + \frac{\pi}{2} &= \frac{5\pi}{4} \\ x &= \frac{5\pi}{4} - \frac{\pi}{2} \\ x &= \frac{3\pi}{4} \end{aligned}$$

Examples

Example 6

Determine the values of $x, 0 \le x \le 3\pi$ so that $y = 4\sin\left[\frac{2}{3}\left(x + \frac{\pi}{2}\right)\right] - 5$ equals -3.

Solution

We want $0 \le x \le 3\pi$ and, from earlier work, we know that the period of $y = 4\sin\left[\frac{2}{3}\left(x + \frac{\pi}{2}\right)\right] - 5$ is 3π.

The solution $x = -\frac{\pi}{4}$ is not in the domain, but the coterminal angle $x = -\frac{\pi}{4} + 3\pi = \frac{11\pi}{4}$ is.

The other solution $x = \frac{3\pi}{4}$ is in the required domain.

Adding or subtracting 3π to either of these solutions takes us outside the domain.

Therefore, the only solutions are $x = \frac{3\pi}{4}$ and $x = \frac{11\pi}{4}$.

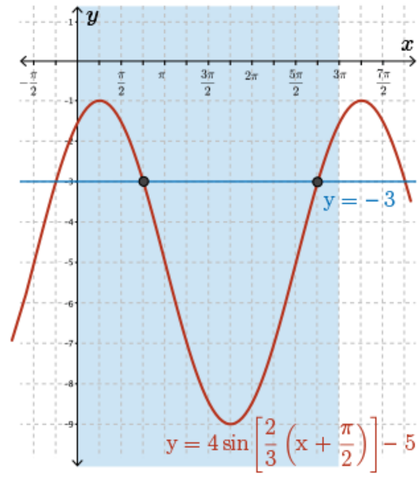
Examples

Example 6

Determine the values of $x, 0 \leq x \leq 3\pi$ so that $y = 4\sin\left[\frac{2}{3}\left(x + \frac{\pi}{2}\right)\right] - 5$ equals -3.

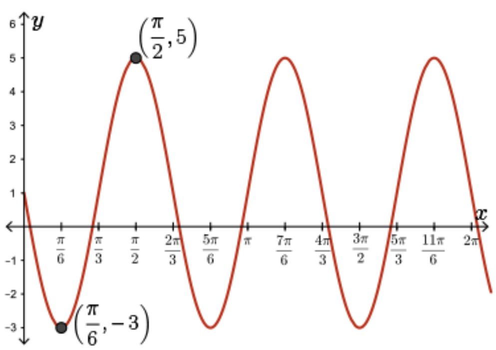
Solution

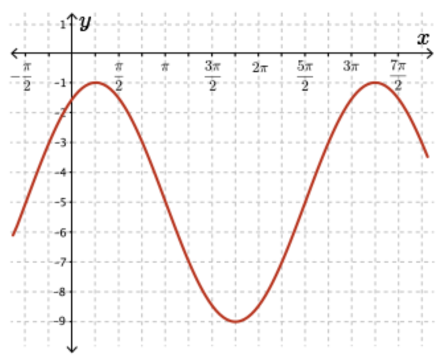
From the equation, we determined that there were two solutions:
$x = \frac{3\pi}{4}$ and $x = \frac{11\pi}{4}$.
Since we have a sketch, we will be able to verify the reliability of our solutions.
The shaded section of the graph contains the part of the function between 0 and 3π radians.
The two points of intersection of the sinusoidal curve with $y = -3$ are shown.



Summary

In this module, we developed equations to model situations which were defined by information given about a function or by a sketch of a function.





We used the graph of the function and the equation to obtain more information about the function.
We used the sketches to read information and we solved some trigonometric equations.
In a future module, we will look at applications which can be modelled with sinusoidal functions.