The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Compound Angle Formulas

In This Module

- We will extend our knowledge of the fundamental trigonometric identities to include the compound angle identities shown here.

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

$$\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$

$$\tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$

These identities involve the sum and difference of two angles and are often referred to as formulas as they provide a means of

- simplifying trigonometric expressions and proving other identities,
- determining exact values for angles related to the acute angles $\frac{\pi}{12}$ or $\frac{5\pi}{12}$ $(15^{\circ}$ or $75^{\circ})$, and
- solving certain trigonometric equations.

Derivation

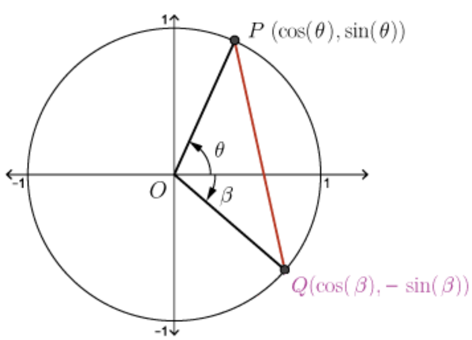
We will begin by deriving the formula for $\cos(A+B)$ using the unit circle.

Consider the two points P and Q on the unit circle, where P is defined by $(\cos(\theta), \sin(\theta))$ for some angle θ, $\theta > 0$ and Q is given by $(\cos(-\beta), \sin(-\beta))$ for some angle β, $\beta > 0$. The coordinates of Q can be simplified to $(\cos(\beta), -\sin(\beta))$.

The measure of $\angle POQ$, with O at the origin, is $\theta + \beta$.

The length of the line segment PQ can be found using

$$|PQ| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$



Thus,

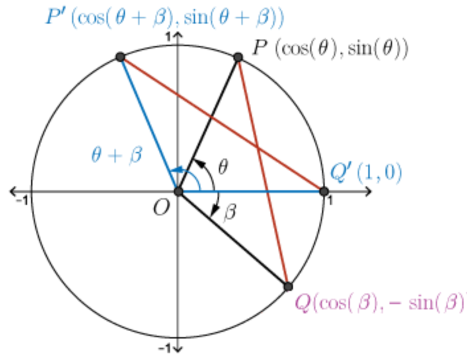
$$\begin{aligned}
|PQ| &= \sqrt{(\cos(\theta)-\cos(\beta))^2 + (\sin(\theta)-(-\sin(\beta)))^2} \\
&= \sqrt{(\cos(\theta)-\cos(\beta))^2 + (\sin(\theta)+\sin(\beta))^2} \\
&= \sqrt{\cos^2(\theta) - 2\cos(\theta)\cos(\beta) + \cos^2(\beta) + \sin^2(\theta) + 2\sin(\theta)\sin(\beta) + \sin^2(\beta)} \\
&= \sqrt{\cos^2(\theta) + \sin^2(\theta) + \cos^2(\beta) + \sin^2(\beta) - 2\cos(\theta)\cos(\beta) + 2\sin(\theta)\sin(\beta)} \\
&= \sqrt{1 + 1 - 2\cos(\theta)\cos(\beta) + 2\sin(\theta)\sin(\beta)} \\
&= \sqrt{2 - 2\cos(\theta)\cos(\beta) + 2\sin(\theta)\sin(\beta)}
\end{aligned}$$

Derivation

Now rotate the points P and Q counterclockwise about the origin by angle of β to obtain $Q'(1, 0)$ and $P'\ (\cos(\theta+\beta), \sin(\theta+\beta))$ on the unit circle, as shown in the diagram.

As a condition of the rotation, $\angle P'OQ' = \theta + \beta$ and $|P'Q'| = |PQ|$.

The length of $P'Q'$ is given by



$$\begin{aligned}|P'Q'| &= \sqrt{(\cos(\theta+\beta)-1)^2 + (\sin(\theta+\beta)-0)^2} \\ &= \sqrt{\cos^2(\theta+\beta) - 2\cos(\theta+\beta) + 1 + \sin^2(\theta+\beta)} \\ &= \sqrt{\cos^2(\theta+\beta) + \sin^2(\theta+\beta) - 2\cos(\theta+\beta) + 1} \\ &= \sqrt{1 - 2\cos(\theta+\beta) + 1} \\ &= \sqrt{2 - 2\cos(\theta+\beta)}\end{aligned}$$

Derivation

Since $|P'Q'| = |PQ|$, we have

$$\begin{aligned}\sqrt{2-2(\cos(\theta+\beta))} &= \sqrt{2 - 2\cos(\theta)\cos(\beta) + 2\sin(\theta)\sin(\beta)} \\ 2 - 2(\cos(\theta+\beta)) &= 2 - 2\cos(\theta)\cos(\beta) + 2\sin(\theta)\sin(\beta) \\ -2(\cos(\theta+\beta)) &= -2\cos(\theta)\cos(\theta) + 2\sin(\theta)\sin(\beta) \\ \cos(\theta+\beta) &= \cos(\theta)\cos(\beta) - \sin(\theta)\sin(\beta)\end{aligned}$$

Angle Sum Formula for Cosine
$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

Derivation

To obtain the formula for $\cos(\theta - \beta)$, we can express it as $\cos(\theta + (-\beta))$ and apply the angle sum formula: $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$.
By setting $A = \theta$ and $B = -\beta$, we have

$$\begin{aligned} \cos(\theta - \beta) &= \cos(\theta + (-\beta)) \\ &= \cos(\theta)\cos(-\beta) - \sin(\theta)\sin(-\beta) \\ &= \cos(\theta)\cos(\beta) - \sin(\theta)(-\sin(\beta)) \\ &= \cos(\theta)\cos(\beta) + \sin(\theta)\sin(\beta) \\ \cos(\theta - \beta) &= \cos(\theta)\cos(\beta) + \sin(\theta)\sin(\beta) \end{aligned}$$

Angle Difference Formula for Cosine
$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$

Angle Sum Formula for Cosine
$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

Derivation

To obtain the formula for $\sin(\theta + \beta)$ we can use the cofunction identities,

$$\sin(x) = \cos\left(\frac{\pi}{2} - x\right) \text{ and } \cos(x) = \sin\left(\frac{\pi}{2} - x\right)$$

and apply the angle difference formula for cosine, $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$.

$$\begin{aligned} \sin(\theta + \beta) &= \cos\left(\frac{\pi}{2} - (\theta + \beta)\right) \\ &= \cos\left(\left(\frac{\pi}{2} - \theta\right) - \beta\right) \end{aligned}$$

If we set $A = \frac{\pi}{2} - \theta$ and $B = \beta$, then

$$\begin{aligned} &= \cos\left(\frac{\pi}{2} - \theta\right)\cos(\beta) + \sin\left(\frac{\pi}{2} - \theta\right)\sin(\beta) \\ &= \sin(\theta)\cos(\beta) + \cos(\theta)\sin(\beta) \\ \sin(\theta + \beta) &= \sin(\theta)\cos(\beta) + \cos(\theta)\sin(\beta) \end{aligned}$$

Angle Sum Formula for Sine
$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

Angle Difference Formula for Sine
$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$

Derivation

To derive the angle sum formula for tangent, we begin by using the quotient identity.

$$\begin{aligned}
\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\
&= \frac{\sin(A)\cos(B)+\cos(A)\sin(B)}{\cos(A)\cos(B)-\sin(A)\sin(B)} \\
&= \frac{(\sin(A)\cos(B)+\cos(A)\sin(B)) \div \cos(A)\cos(B)}{(\cos(A)\cos(B)-\sin(A)\sin(B)) \div \cos(A)\cos(B)} \\
&= \frac{\dfrac{\sin(A)\,\cancel{\cos(B)}}{\cos(A)\,\cancel{\cos(B)}} + \dfrac{\cancel{\cos(A)}\,\sin(B)}{\cancel{\cos(A)}\,\cos(B)}}{\cancel{\dfrac{\cos(A)\cos(B)}{\cos(A)\cos(B)}} - \dfrac{\sin(A)\sin(B)}{\cos(A)\cos(B)}} \\
\tan(A+B) &= \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}
\end{aligned}$$

Angle Sum Formula for Tangent

$$\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$

Derivation

Angle Difference Formula for Tangent

$$\tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$

Summary

Angle Sum Formulas

$$\sin(A+B) = \sin(A)\cos(B)+\cos(A)\sin(B)$$

$$\cos(A+B) = \cos(A)\cos(B)-\sin(A)\sin(B)$$

$$\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$

Angle Difference Formulas

$$\sin(A-B) = \sin(A)\cos(B)-\cos(A)\sin(B)$$

$$\cos(A-B) = \cos(A)\cos(B)+\sin(A)\sin(B)$$

$$\tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}$$

Examples

Example 1

Determine an equivalent trigonometric expression using an appropriate compound angle formula.

a. $\sin\left(x - \frac{\pi}{2}\right)$ **b.** $\cos\left(x + \frac{3\pi}{2}\right)$ **c.** $\tan\left(x + \frac{5\pi}{6}\right)$

Solution

a. Using the angle difference formula,

$$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

By setting $A = x$ and $B = \frac{\pi}{2}$, we have

$$\begin{aligned}\sin\left(x - \frac{\pi}{2}\right) &= \sin(x)\cos\left(\frac{\pi}{2}\right) - \cos(x)\sin\left(\frac{\pi}{2}\right)\\ &= \sin(x)(0) - \cos(x)(1)\\ &= -\cos(x)\end{aligned}$$

Previously, we may have argued that

$$\begin{aligned}\sin\left(x - \frac{\pi}{2}\right) &= \sin\left(-\left(\frac{\pi}{2} - x\right)\right)\\ &= -\sin\left(\frac{\pi}{2} - x\right)\\ &= -\cos(x)\end{aligned}$$

Examples

Example 1

Determine an equivalent trigonometric expression using an appropriate compound angle formula.

a. $\sin\left(x - \frac{\pi}{2}\right)$ **b.** $\cos\left(x + \frac{3\pi}{2}\right)$ **c.** $\tan\left(x + \frac{5\pi}{6}\right)$

Solution

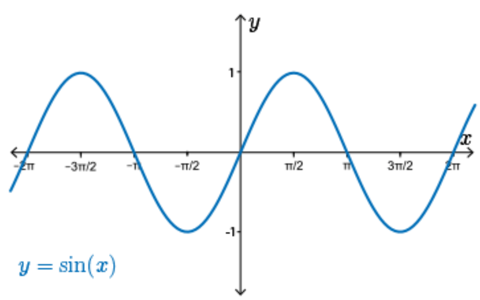
b. Using the angle sum formula

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

we have

$$\begin{aligned}\cos\left(x + \frac{3\pi}{2}\right) &= \cos(x)\cos\left(\frac{3\pi}{2}\right) - \sin(x)\sin\left(\frac{3\pi}{2}\right)\\ &= \cos(x)(0) - \sin(x)(-1)\\ &= \sin(x)\end{aligned}$$

To verify this equivalence graphically, translate $y = \cos(x)$ to the left $\frac{3\pi}{2}$ units. The image curve is the sine curve.



Examples

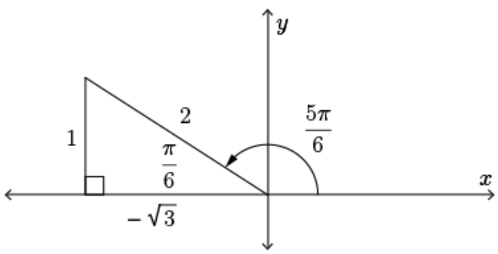
Example 1

Determine an equivalent trigonometric expression using an appropriate compound angle formula.

a. $\sin\left(x - \frac{\pi}{2}\right)$ **b.** $\cos\left(x + \frac{3\pi}{2}\right)$ **c.** $\tan\left(x + \frac{5\pi}{6}\right)$

Solution

c. Using $\tan(A+B) = \dfrac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$ where $A = x$ and $B = \frac{5\pi}{6}$, we have



$$\begin{aligned}\tan\left(x + \frac{5\pi}{6}\right) &= \frac{\tan(x) + \tan\left(\frac{5\pi}{6}\right)}{1 - \tan(x)\tan\left(\frac{5\pi}{6}\right)} \\ &= \frac{\tan(x) - \frac{1}{\sqrt{3}}}{1 - \tan(x)\left(-\frac{1}{\sqrt{3}}\right)} \\ &= \frac{\left(\tan(x) - \frac{1}{\sqrt{3}}\right)}{\left(1 + \frac{\tan(x)}{\sqrt{3}}\right)} \times \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{\sqrt{3}\tan(x) - 1}{\sqrt{3} + \tan(x)}\end{aligned}$$

Examples

Example 2

Express as a single trigonometric ratio using an appropriate compound angle formula.

a. $\cos(\theta)\cos(2\theta) - \sin(\theta)\sin(2\theta)$ **b.** $\dfrac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{3\pi}{4}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{3\pi}{4}\right)}$ **c.** $\frac{1}{2}\cos(x) + \frac{\sqrt{3}}{2}\sin(x)$

Solution

a. Using the sum formula for cosine,

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

where $A = \theta$ and $B = 2\theta$, we have

$$\begin{aligned}\cos(\theta)\cos(2\theta) - \sin(\theta)\sin(2\theta) &= \cos(\theta + 2\theta) \\ &= \cos(3\theta)\end{aligned}$$

b. Using the difference formula for tangent,

$$\tan(A-B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$$

where $A = \frac{\pi}{3}$ and $B = \frac{3\pi}{4}$, we have

$$\begin{aligned}\frac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{3\pi}{4}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{3\pi}{4}\right)} &= \tan\left(\frac{\pi}{3} - \frac{3\pi}{4}\right) \\ &= \tan\left(\frac{4\pi}{12} - \frac{9\pi}{12}\right) \\ &= \tan\left(-\frac{5\pi}{12}\right) \\ &= -\tan\left(\frac{5\pi}{12}\right)\end{aligned}$$

Examples

Example 2

Express as a single trigonometric ratio using an appropriate compound angle formula.

a. $\cos(\theta)\cos(2\theta) - \sin(\theta)\sin(2\theta)$

b. $\dfrac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{3\pi}{4}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{3\pi}{4}\right)}$

c. $\dfrac{1}{2}\cos(x) + \dfrac{\sqrt{3}}{2}\sin(x)$

Solution

c. Since $\sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$ and $\cos\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$ then,

$$\begin{aligned}\frac{1}{2}\cos(x) + \frac{\sqrt{3}}{2}\sin(x) &= \sin\left(\frac{\pi}{6}\right)\cos(x) + \cos\left(\frac{\pi}{6}\right)\sin(x)\\ &= \sin\left(\frac{\pi}{6} + x\right)\end{aligned}$$

Examples

Example 2

Express as a single trigonometric ratio using an appropriate compound angle formula.

a. $\cos(\theta)\cos(2\theta) - \sin(\theta)\sin(2\theta)$

b. $\dfrac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{3\pi}{4}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{3\pi}{4}\right)}$

c. $\dfrac{1}{2}\cos(x) + \dfrac{\sqrt{3}}{2}\sin(x)$

Solution

c. An alternate expression can be obtained using $\sin\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$ and $\cos\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2}$.

$$\begin{aligned}\frac{1}{2}\cos(x) + \frac{\sqrt{3}}{2}\sin(x) &= \cos\left(\frac{\pi}{3}\right)\cos(x) + \sin\left(\frac{\pi}{3}\right)\sin(x)\\ &= \cos\left(\frac{\pi}{3} - x\right)\end{aligned}$$

This implies that $\sin\left(\dfrac{\pi}{6} + x\right) = \cos\left(\dfrac{\pi}{3} - x\right)$.

Examples

Example 2

Express as a single trigonometric ratio using an appropriate compound angle formula.

a. $\cos(\theta)\cos(2\theta) - \sin(\theta)\sin(2\theta)$

b. $\dfrac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{3\pi}{4}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{3\pi}{4}\right)}$

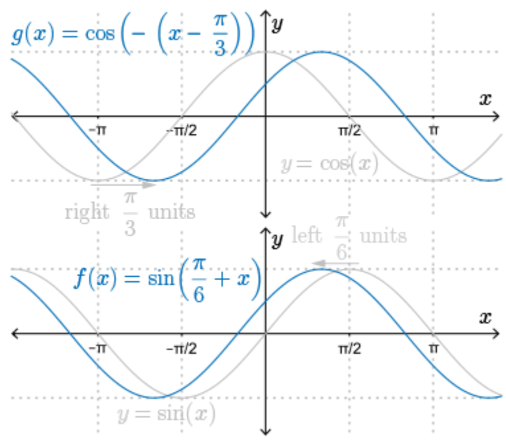
c. $\dfrac{1}{2}\cos(x) + \dfrac{\sqrt{3}}{2}\sin(x)$

Solution

To verify $\sin\left(\dfrac{\pi}{6} + x\right) = \cos\left(\dfrac{\pi}{3} - x\right)$ graphically, let $f(x) = \sin\left(\dfrac{\pi}{6} + x\right)$ and $g(x) = \cos\left(\dfrac{\pi}{3} - x\right)$.

The graph of $f(x) = \sin\left(x + \dfrac{\pi}{6}\right)$ can be obtained by translating the graph of $y = \sin(x)$ to the left $\dfrac{\pi}{6}$.

The graph of $g(x) = \cos\left(-\left(x - \dfrac{\pi}{3}\right)\right)$ can be obtained by reflecting the graph of $y = \cos(x)$ in the y-axis, and translating the curve to the right $\dfrac{\pi}{3}$.



Examples

We can use compound angle formulas to determine the exact value of any angle corresponding to the reference angles $15°$ and $75°$, or in radians, $\frac{\pi}{12}$ and $\frac{5\pi}{12}$.

Example 3

Determine the exact value of each using a compound angle formula.

a. $\sin\left(\dfrac{13\pi}{12}\right)$

b. $\cos(195°)$

Solution

a. $\sin\left(\dfrac{13\pi}{12}\right)$

To determine the exact value of $\sin\left(\dfrac{13\pi}{12}\right)$, we express $\dfrac{13\pi}{12}$ as a sum or difference of two angles corresponding to the related acute angles $\dfrac{\pi}{6}$, $\dfrac{\pi}{4}$, or $\dfrac{\pi}{3}$. For example,

$$\begin{aligned}\frac{13\pi}{12} &= \frac{9\pi}{12} + \frac{4\pi}{12} \\ &= \frac{3\pi}{4} + \frac{\pi}{3}\end{aligned}$$

So, $\sin\left(\dfrac{13\pi}{12}\right) = \sin\left(\dfrac{3\pi}{4} + \dfrac{\pi}{3}\right)$.

Examples

We can use compound angle formulas to determine the exact value of any angle corresponding to the reference angles 15° and 75°, or in radians, $\frac{\pi}{12}$ and $\frac{5\pi}{12}$.

Example 3

Determine the exact value of each using a compound angle formula.

a. $\sin\left(\frac{13\pi}{12}\right)$

b. $\cos(195^\circ)$

Solution

$$\begin{aligned}
\sin\left(\frac{13\pi}{12}\right) &= \sin\left(\frac{3\pi}{4}+\frac{\pi}{3}\right)\\
&= \sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)+\cos\left(\frac{3\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)\\
&= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)+\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)\\
&= \frac{1-\sqrt{3}}{2\sqrt{2}}\\
&= \frac{1-\sqrt{3}}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{4}
\end{aligned}$$

Therefore, the exact value of $\sin\left(\frac{13\pi}{12}\right)$ is

$$\frac{\sqrt{2}-\sqrt{6}}{4}$$

Examples

We can use compound angle formulas to determine the exact value of any angle corresponding to the reference angles 15° and 75°, or in radians, $\frac{\pi}{12}$ and $\frac{5\pi}{12}$.

Example 3

Determine the exact value of each using a compound angle formula.

a. $\sin\left(\frac{13\pi}{12}\right)$

b. $\cos(195^\circ)$

Solution

b. $\cos(195^\circ)$

Since $195^\circ = 225^\circ - 30^\circ$

$$\begin{aligned}
\cos(195^\circ) &= \cos(225^\circ - 30^\circ)\\
&= \cos(225^\circ)\cos(30^\circ)+\sin(225^\circ)\sin(30^\circ)\\
&= \left(-\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{\sqrt{3}}{2}\right)+\left(-\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{2}\right)\\
&= \frac{-\sqrt{3}-1}{2\sqrt{2}}\\
&= \frac{-\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

Since $195^\circ = 135^\circ + 60^\circ$

$$\begin{aligned}
\cos(195^\circ) &= \cos(135^\circ + 60^\circ)\\
&= \cos(135^\circ)\cos(60^\circ)-\sin(135^\circ)\sin(60^\circ)\\
&= \left(-\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{2}\right)-\left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{\sqrt{3}}{2}\right)\\
&= \frac{-1-\sqrt{3}}{2\sqrt{2}}\\
&= \frac{-\sqrt{2}-\sqrt{6}}{4}
\end{aligned}$$

Examples

Example 4

Prove that $\cos(x+y)\cos(x-y) = \cos^2(x) - \sin^2(y)$.

Solution

$$\begin{aligned}
\text{L.S.} &= \cos(x+y)\cos(x-y) \\
&= \Big(\cos(x)\cos(y) - \sin(x)\sin(y)\Big)\Big(\cos(x)\cos(y) + \sin(x)\sin(y)\Big) \\
&= \Big(\cos(x)\cos(y)\Big)^2 + \cancel{\cos(x)\cos(y)\sin(x)\sin(y)} - \cancel{\sin(x)\sin(y)\cos(x)\cos(y)} - \Big(\sin(x)\sin(y)\Big)^2 \\
&= \Big(\cos(x)\cos(y)\Big)^2 - \Big(\sin(x)\sin(y)\Big)^2 \\
&= \cos^2(x)\cos^2(y) - \sin^2(x)\sin^2(y) \\
&= \cos^2(x)\Big(1-\sin^2(y)\Big) - \Big(1-\cos^2(x)\Big)\sin^2(y) \\
&= \cos^2(x) - \cancel{\cos^2(x)\sin^2(y)} - \sin^2(y) + \cancel{\cos^2(x)\sin^2(y)} \\
&= \cos^2(x) - \sin^2(y) \\
&= \text{R.S.}
\end{aligned}$$

Therefore, $\cos(x+y)\cos(x-y) = \cos^2(x) - \sin^2(y)$.

Examples

Example 5

Solve for x in $\dfrac{1}{\sin\left(x-\frac{\pi}{6}\right) - \sin\left(x+\frac{\pi}{6}\right)} = \sqrt{2}$, where $0 \le x \le 2\pi$.

Solution

$$\begin{aligned}
\frac{1}{\sin\left(x-\frac{\pi}{6}\right) - \sin\left(x+\frac{\pi}{6}\right)} &= \sqrt{2} \\
\frac{1}{\left(\sin(x)\cos\left(\frac{\pi}{6}\right) - \cos(x)\sin\left(\frac{\pi}{6}\right)\right) - \left(\sin(x)\cos\left(\frac{\pi}{6}\right) + \cos(x)\sin\left(\frac{\pi}{6}\right)\right)} &= \sqrt{2} \\
\frac{1}{\sin(x)\cos\left(\frac{\pi}{6}\right) - \cos(x)\sin\left(\frac{\pi}{6}\right) - \sin(x)\cos\left(\frac{\pi}{6}\right) - \cos(x)\sin\left(\frac{\pi}{6}\right)} &= \sqrt{2} \\
\frac{1}{-\cos(x)\sin\left(\frac{\pi}{6}\right) - \cos(x)\sin\left(\frac{\pi}{6}\right)} &= \sqrt{2} \\
\frac{1}{-2\cos(x)\sin\left(\frac{\pi}{6}\right)} &= \sqrt{2}
\end{aligned}$$

Examples

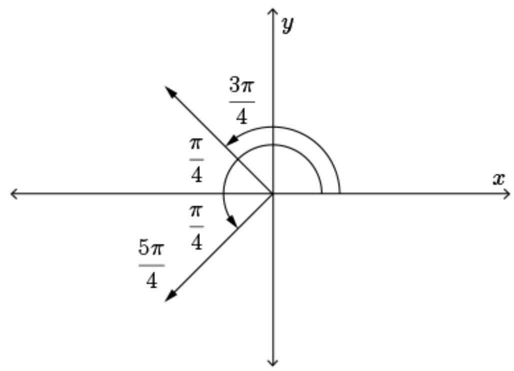
Example 5

Solve for x in $\dfrac{1}{\sin\left(x-\frac{\pi}{6}\right)-\sin\left(x+\frac{\pi}{6}\right)}=\sqrt{2}$, where $0 \le x \le 2\pi$.

Solution

$$\begin{aligned}\frac{1}{-2\cos(x)\sin\left(\frac{\pi}{6}\right)} &= \sqrt{2}\\ \frac{1}{-2\cos(x)\left(\frac{1}{2}\right)} &= \sqrt{2}\\ \frac{1}{-\cos(x)} &= \sqrt{2}\\ -\sqrt{2}\cos(x) &= 1\\ \cos(x) &= -\frac{1}{\sqrt{2}}\end{aligned}$$

Therefore, $x=\frac{3\pi}{4}$ or $\frac{5\pi}{4}$.



Examples

Example 6

Solve $\sqrt{3}\sin(x)+3\cos(x)=-\sqrt{6}, 0\le x\le 2\pi$ by first expressing $\sqrt{3}\sin(x)+3\cos(x)$ in the form $a\sin(x-h)$.

Solution

First, find a value for a and h such that $\sqrt{3}\sin(x)+3\cos(x)=a\sin(x-h)$.

$$\begin{aligned}\sqrt{3}\sin(x)+3\cos(x) &= a\sin(x-h)\\ &= a\Big(\sin(x)\cos(h)-\cos(x)\sin(h)\Big)\\ &= a\sin(x)\cos(h)-a\cos(x)\sin(h)\\ \sqrt{3}\sin(x)+3\cos(x) &= a\sin(x)\cos(h)-a\cos(x)\sin(h)\end{aligned}$$

Therefore,

$$a\cos(h)=\sqrt{3} \quad (1)$$
$$-a\sin(h)=3 \quad (2)$$

Examples

Example 6

Solve $\sqrt{3}\sin(x) + 3\cos(x) = -\sqrt{6}, 0 \le x \le 2\pi$ by first expressing $\sqrt{3}\sin(x) + 3\cos(x)$ in the form $a\sin(x - h)$.

Solution

We have established two equations to solve for the two unknowns a and h,

$$a\cos(h) = \sqrt{3} \quad (1)$$

$$-a\sin(h) = 3 \quad (2)$$

Dividing equation (2) by equation (1),

$$\frac{-a\sin(h)}{a\cos(h)} = \frac{3}{\sqrt{3}}$$

$$-\tan(h) = \frac{3}{\sqrt{3}}$$

$$\tan(h) = -\frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$\tan(h) = -\sqrt{3}$$

and therefore, $h = \dfrac{2\pi}{3}$ is one possible solution.

Examples

Example 6

Solve $\sqrt{3}\sin(x) + 3\cos(x) = -\sqrt{6}, 0 \le x \le 2\pi$ by first expressing $\sqrt{3}\sin(x) + 3\cos(x)$ in the form $a\sin(x - h)$.

Solution

We need only one value for h; substituting $h = \dfrac{2\pi}{3}$ into (1),

$$a\cos\left(\frac{2\pi}{3}\right) = \sqrt{3}$$

$$a\left(-\frac{1}{2}\right) = \sqrt{3}$$

$$a = -2\sqrt{3}$$

Therefore, $\sqrt{3}\sin(x) + 3\cos(x) = -2\sqrt{3}\sin\left(x - \dfrac{2\pi}{3}\right)$.

Substituting this form into $\sqrt{3}\sin(x) + 3\cos(x) = -\sqrt{6}$, we have

$$-2\sqrt{3}\sin\left(x - \frac{2\pi}{3}\right) = -\sqrt{6}$$

Examples

Example 6

Solve $\sqrt{3}\sin(x)+3\cos(x)=-\sqrt{6}, 0\le x\le 2\pi$ by first expressing $\sqrt{3}\sin(x)+3\cos(x)$ in the form $a\sin(x-h)$.

Solution

$$-2\sqrt{3}\sin\left(x-\frac{2\pi}{3}\right)=-\sqrt{6}$$

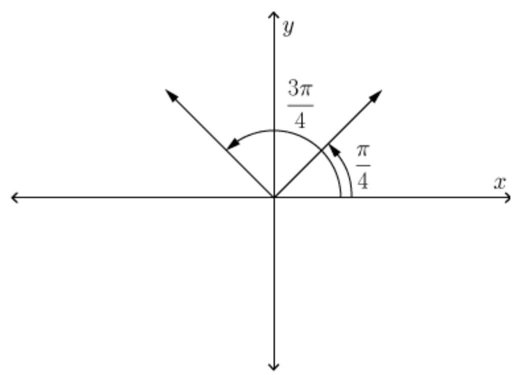
We are now ready to solve for x by first isolating $\sin\left(x-\frac{2\pi}{3}\right)$.

$$\sin\left(x-\frac{2\pi}{3}\right)=-\frac{\sqrt{6}}{-2\sqrt{3}}$$

$$\sin\left(x-\frac{2\pi}{3}\right)=\frac{\sqrt{2}}{2} \text{ or } \frac{1}{\sqrt{2}}$$

$$x-\frac{2\pi}{3}=\frac{\pi}{4},\frac{3\pi}{4}$$

All possible solutions for $x-\frac{2\pi}{3}$ are given by $\frac{\pi}{4}+2\pi n$ and $\frac{3\pi}{4}+2\pi n$ where $n\in\mathbb{Z}$.



Examples

Example 6

Solve $\sqrt{3}\sin(x)+3\cos(x)=-\sqrt{6}, 0\le x\le 2\pi$ by first expressing $\sqrt{3}\sin(x)+3\cos(x)$ in the form $a\sin(x-h)$.

Solution

For $0\le x\le 2\pi$, we have

$$\begin{aligned}x-\frac{2\pi}{3}&=\frac{\pi}{4},\frac{3\pi}{4}\\ x&=\frac{\pi}{4}+\frac{2\pi}{3},\frac{3\pi}{4}+\frac{2\pi}{3}\\ x&=\frac{3\pi}{12}+\frac{8\pi}{12},\frac{9\pi}{12}+\frac{8\pi}{12}\\ x&=\frac{11\pi}{12},\frac{17\pi}{12}\end{aligned}$$

Therefore, the solutions to $\sqrt{3}\sin(x)+3\cos(x)=-\sqrt{6}$, for $0\le x\le 2\pi$ are $\frac{11\pi}{12},\frac{17\pi}{12}$.

Examples

Example 6

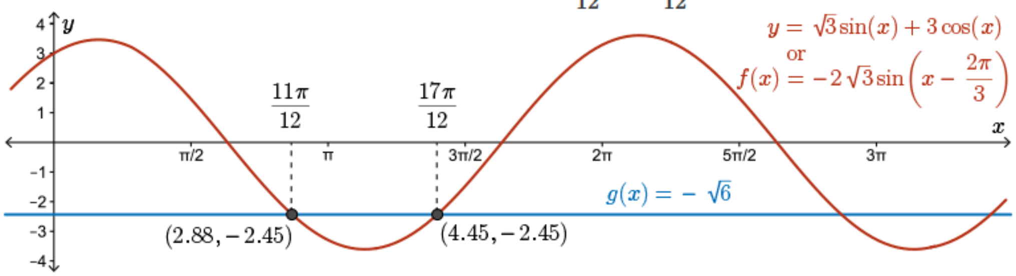
Solve $\sqrt{3}\sin(x) + 3\cos(x) = -\sqrt{6}, 0 \leq x \leq 2\pi$ by first expressing $\sqrt{3}\sin(x) + 3\cos(x)$ in the form $a\sin(x - h)$.

Solution

We were able to solve $\sqrt{3}\sin(x) + 3\cos(x) = -\sqrt{6}$ by first determining an equivalent expression for $\sqrt{3}\sin(x) + 3\cos(x)$ using an appropriate compound angle formula.

We can verify this equivalence and the solutions to the equation using graphing technology.

The graph of $y = \sqrt{3}\sin(x) + 3\cos(x)$ is the same as the graph of $y = -2\sqrt{3}\sin\left(x - \frac{2\pi}{3}\right)$ and this graph intersects the line $y = -\sqrt{6}$ at approximately 2.88 and 4.45, or $\frac{11\pi}{12}$ and $\frac{17\pi}{12}$.



Summary

Angle Sum Formulas

$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

$\tan(A + B) = \dfrac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$

Angle Difference Formulas

$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$

$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$

$\tan(A - B) = \dfrac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$

Compound angle formulas can be used to

- simplify trigonometric expressions and determine equivalent forms,
- prove identities,
- find exact values for angles related to the acute angles $\frac{\pi}{12}$ and $\frac{5\pi}{12}$, and
- solve trigonometric equations.