Equivalent Trigonometric Expressions

In This Module

- We will review and examine the basic fundamental relationships between trigonometric functions.
- We will identify equivalent trigonometric expressions, demonstrating equivalence algebraically and graphically.
- We will determine the non-permissible values of the variable involved in trigonometric expressions and equations.

In This Module

Definition of an Identity

A mathematical **identity** is an equation or statement that is true for all permissible values of the variables in the equation.

For example,

$$(x+y)^2 = x^2 + 2xy + y^2$$

is a mathematical statement that is true for all values of x and y. Thus, it is an identity.
This is in contrast to *conditional* equations such as "$x^2 - 5x + 6 = 0$", which are only true for specific values of the variable, x. In this case, x must equal 2 or 3 for the statement to be true.

Fundamental Trigonometric Identities

Let's recall some fundamental trigonometric identities, you should be familiar with, from the previous unit on trigonometric functions.

The reciprocal identities, by definition of the functions, are

$$\text{Cosecant: } \csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\text{Secant: } \sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\text{Cotangent: } \cot(\theta) = \frac{1}{\tan(\theta)}$$

The quotient identities are

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \quad \text{and} \quad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

The Pythagorean identity is

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

The quotient and Pythagorean identities were developed previously, using the unit circle, in the lesson "Trigonometric Ratios and Special Triangles" of the Trigonometric Functions unit.

Fundamental Trigonometric Identities

Equivalent Identities

The Pythagorean identity, $\sin^2(\theta) + \cos^2(\theta) = 1$, can be expressed in various forms.

By simply rearranging the terms, we have

$$\sin^2(\theta) = 1 - \cos^2(\theta) \text{ and } \cos^2(\theta) = 1 - \sin^2(\theta)$$

By dividing each side of the equation by $\sin^2(\theta)$, we obtain

$$\frac{\sin^2(\theta)}{\sin^2(\theta)} + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

Similarly, dividing by $\cos^2(\theta)$,

$$\frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

The three Pythagorean identities are

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

Fundamental Trigonometric Identities

Reciprocal identities

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)}$$
$$\cot(\theta) = \frac{1}{\tan(\theta)}$$

Quotient identities

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \text{ and } \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

Pythagorean identities

$$\sin^2(\theta) + \cos^2(\theta) = 1$$
$$1 + \cot^2(\theta) = \csc^2(\theta)$$
$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

Each of these fundamental identities, with the exception of $\sin^2(\theta) + \cos^2(\theta) = 1$, have non-permissible values for the variable, θ.

First, the trigonometric functions within the equations have restrictions on their domain and secondly, the denominator of any rational trigonometric expression cannot equal zero.

Fundamental Trigonometric Identities

For example, consider $\csc(\theta) = \frac{1}{\sin(\theta)}$. For certain values of θ, the expression on each side of the equation is undefined.

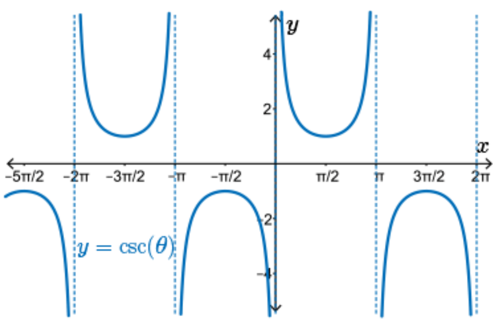
The domain of the function $y = \csc(\theta)$ is the set of all real values of θ such that $\theta \neq n\pi, n \in \mathbb{Z}$.

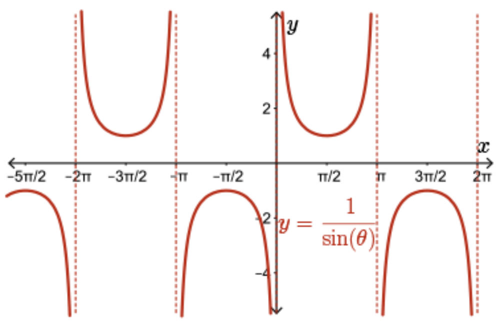
The expression $\frac{1}{\sin(\theta)}$ is undefined when $\sin(\theta) = 0$; that is, when θ is $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

So, non-permissible values of θ are $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ Thus, $\theta \neq n\pi, n \in \mathbb{Z}$.

$$\csc(\theta) = \frac{1}{\sin(\theta)}, \theta \neq n\pi, n \in \mathbb{Z}$$

Graphically, non-permissible values result in vertical asymptotes on the graphs of $y = \csc(\theta)$ and $y = \frac{1}{\sin(\theta)}$.





Examples

Example 1

Simplify $\dfrac{1+\cot(x)}{\csc(x)}$, identifying any non-permissible values of the variable.

Solution

Express each trigonometric ratio in terms of $\sin(x)$ and $\cos(x)$.

$$\begin{aligned}\frac{1+\cot(x)}{\csc(x)} &= \frac{1+\frac{\cos(x)}{\sin(x)}}{\frac{1}{\sin(x)}} \\ &= \left(1+\frac{\cos(x)}{\sin(x)}\right)\times\frac{\sin(x)}{1} \\ &= 1(\sin(x))+\left(\frac{\cos(x)}{\sin(x)}\right)(\sin(x)) \\ &= \sin(x)+\cos(x)\end{aligned}$$

Thus, $\dfrac{1+\cot(x)}{\csc(x)}=\sin(x)+\cos(x)$.

Examples

Example 1

Simplify $\dfrac{1+\cot(x)}{\csc(x)}$, identifying any non-permissible values of the variable.

Solution

To determine the non-permissible values of x in the expression $\dfrac{1+\cot(x)}{\csc(x)}$, we must identify when $\cot(x)$ is undefined and when $\csc(x)$ is zero or undefined.

Non-permissible values may be more easily identified from the non-simplified equivalent expression

$$\frac{1+\cot(x)}{\csc(x)} = \frac{1+\frac{\cos(x)}{\sin(x)}}{\frac{1}{\sin(x)}}$$

We see that $\sin(x)\neq 0$ so $x\neq n\pi, n\in\mathbb{Z}$.

Note that $\csc(x)$, that is $\dfrac{1}{\sin(x)}$, will never equal zero.

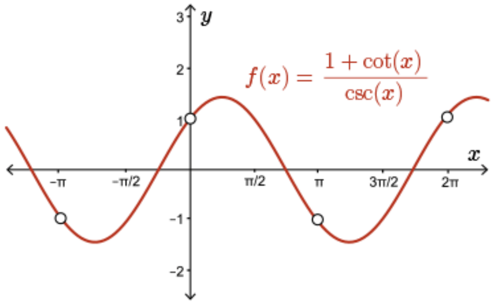
Therefore, $\dfrac{1+\cot(x)}{\csc(x)}=\sin(x)+\cos(x)$ for all real values of $x, x\neq n\pi, n\in\mathbb{Z}$.

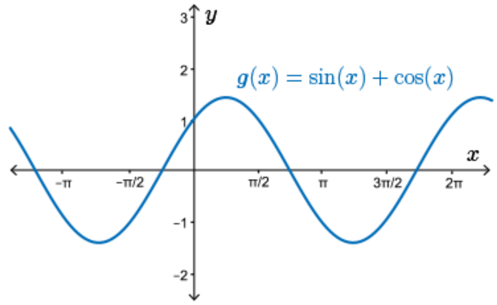
Examples

This equivalence can be demonstrated by graphing the two functions

$$f(x) = \frac{1+\cot(x)}{\csc(x)} \text{ and } g(x) = \sin(x) + \cos(x)$$

using graphing technology.





The graphs of these two functions appear to be identical; however, the domain of each function is not the same.
The domain of $g(x) = \sin(x) + \cos(x)$ is $\{x \mid x \in \mathbb{R}\}$.

From our discussion of the non-permissible values, we know that the domain of $f(x) = \frac{1+\cot(x)}{\csc(x)}$ must be $\{x \mid x \neq n\pi, x \in \mathbb{R}, n \in \mathbb{Z}\}$.
There are, in fact, holes in the graph of this function at these non-permissible values.

Examples

Example 2

Simplify $\frac{1}{\sin(\theta)\cos(\theta)} - \tan(\theta)$, identifying any non-permissible values on the variable.

Solution

$$\frac{1}{\sin(\theta)\cos(\theta)} - \tan(\theta) = \frac{1}{\sin(\theta)\cos(\theta)} - \frac{\sin(\theta)}{\cos(\theta)} \quad \text{using a quotient identity}$$

$$= \frac{1}{\sin(\theta)\cos(\theta)} - \frac{\sin(\theta)\sin(\theta)}{\cos(\theta)\sin(\theta)} \quad \text{finding a common denominator}$$

$$= \frac{1-\sin^2(\theta)}{\sin(\theta)\cos(\theta)}$$

$$= \frac{\cos^2(\theta)}{\sin(\theta)\cos(\theta)} \quad \text{using a Pythagorean identity}$$

$$= \frac{\cos^{\not 2}(\theta)}{\sin(\theta)\not{\cos(\theta)}}$$

$$= \frac{\cos(\theta)}{\sin(\theta)} \quad \text{simplify the expression}$$

$$= \cot(\theta) \quad \text{using a quotient identity}$$

Examples

Example 2

Simplify $\dfrac{1}{\sin(\theta)\cos(\theta)} - \tan(\theta)$, identifying any non-permissible values on the variable.

Solution

$$\frac{1}{\sin(\theta)\cos(\theta)} - \tan(\theta) = \frac{1}{\sin(\theta)\cos(\theta)} - \frac{\sin(\theta)}{\cos(\theta)}$$

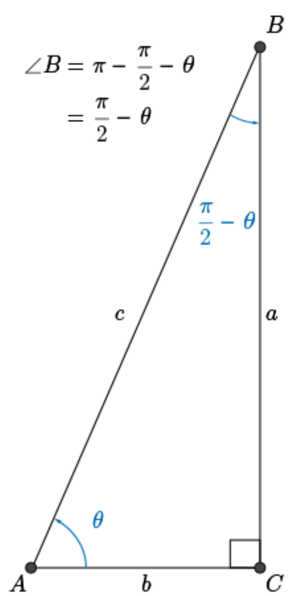
Using the non-simplified equivalent form of the expression to help identify the non-permissible values of the variable θ, we see that the expression is defined when $\sin(\theta)$ and $\cos(\theta)$ are not equal to zero.

Thus, $\theta \neq n\pi, n \in \mathbb{Z}$ where $\sin(\theta) = 0$ and $\theta \neq \dfrac{\pi}{2} + n\pi, n \in \mathbb{Z}$ where $\cos(\theta) = 0$. Simplifying, we have $\theta \neq \dfrac{n\pi}{2}, n \in \mathbb{Z}$. Notice that this covers the restrictions on θ for $\cot(\theta)$, the simplified form of this expression.

Therefore, $\dfrac{1}{\sin(\theta)\cos(\theta)} - \tan(\theta) = \cot(\theta)$ for all real $\theta, \theta \neq \dfrac{n\pi}{2}, n \in \mathbb{Z}$.

Cofunction Identities

In the right triangle ABC, if $\angle BAC = \theta$ then, by the sum of the angles of a triangle, $\angle ABC = \dfrac{\pi}{2} - \theta$.



Since $\sin(\theta) = \dfrac{a}{c}$ and $\cos\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{a}{c}$, then

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

Similarly, $\cos(\theta) = \dfrac{b}{c}$ and $\sin\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{b}{c}$.

Therefore,

$$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$$

Also, $\tan(\theta) = \dfrac{a}{b}$ and $\cot\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{a}{b}$.

Therefore,

$$\tan(\theta) = \cot\left(\frac{\pi}{2} - \theta\right)$$

Cofunction Identities

Summary of Cofunction Identities

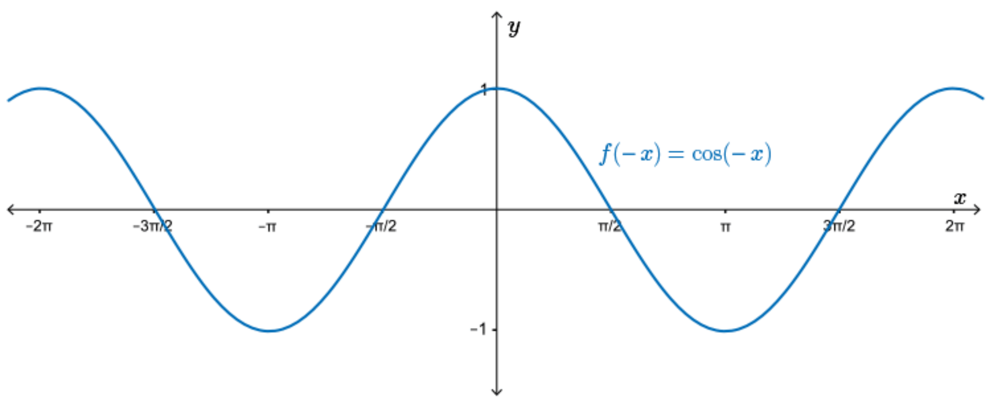
$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$	$\tan(\theta) = \cot\left(\frac{\pi}{2} - \theta\right)$
$\csc(\theta) = \sec\left(\frac{\pi}{2} - \theta\right)$	$\sec(\theta) = \csc\left(\frac{\pi}{2} - \theta\right)$	$\cot(\theta) = \tan\left(\frac{\pi}{2} - \theta\right)$

These identities are called cofunction identities since they show a relationship between sine and cosine and a relationship between tangent and cotangent.
The value of one function at an angle is equal to the value of the cofunction at the complement of the angle.
For example, $\sin(10^\circ) = \cos(80^\circ)$ and $\tan\left(\frac{\pi}{3}\right) = \cot\left(\frac{\pi}{6}\right)$.
This group of identities extends to include the reciprocal functions.

Equivalent Expressions from Symmetry

$f(x) = \cos(x)$

The function $f(x) = \cos(x)$ is an even function and is symmetric about the y-axis.

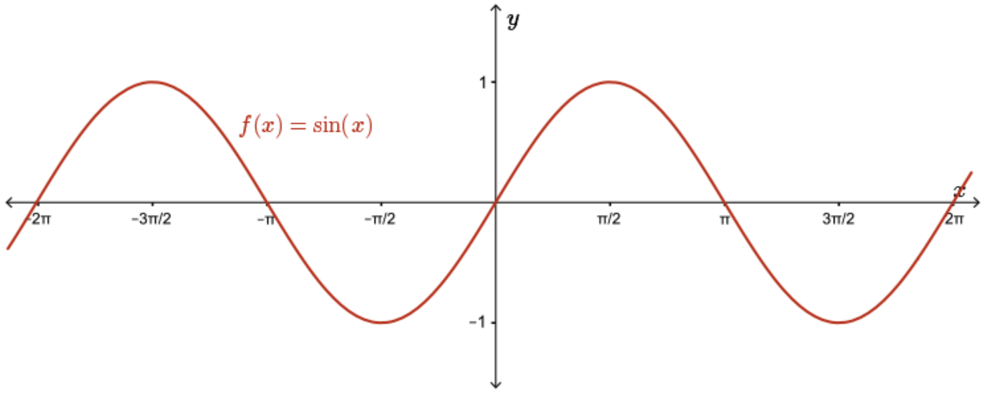


This means that $f(-x) = f(x)$; thus, $\cos(-x) = \cos(x)$.

Equivalent Expressions from Symmetry

$f(x) = \sin(x)$

The function $f(x) = \sin(x)$ is an odd function, symmetric about the origin.
If we reflect $f(x)$ in the x-axis, followed by a reflection in the y-axis, the image will be the same as $f(x)$.

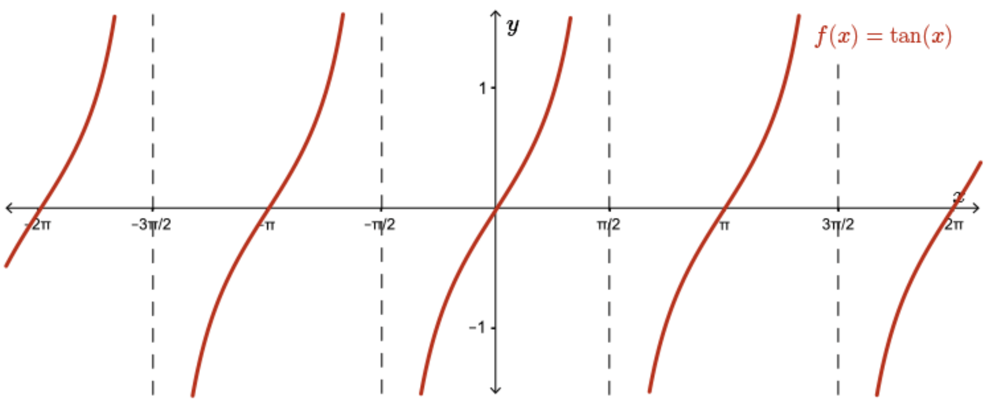


This means that $-f(-x) = f(x)$ or $f(-x) = -f(x)$; thus, $\sin(-x) = -\sin(x)$.

Equivalent Expressions from Symmetry

$f(x) = \tan(x)$

Similarly, the function $f(x) = \tan(x)$ is an odd function, so $\tan(-x) = -\tan(x)$.



Equivalent Expressions from Symmetry

Even Symmetry

$\cos(-\theta) = \cos(\theta)$

$\sec(-\theta) = \sec(\theta)$

Odd Symmetry

$\sin(-\theta) = -\sin(\theta)$

$\tan(-\theta) = -\tan(\theta)$

$\csc(-\theta) = -\csc(\theta)$

$\cot(-\theta) = -\cot(\theta)$

From our studies of reciprocal functions, we know $y = \dfrac{1}{f(x)}$ is even if $y = f(x)$ is even and $y = \dfrac{1}{f(x)}$ is odd if $y = f(x)$ is odd. We can extend this list of identities to include the reciprocal trigonometric functions.

Examples

Example 3

Express $\sec\left(\theta - \dfrac{\pi}{2}\right)$ in terms of $\sin(\theta)$.

Solution

$$\begin{aligned}
\sec\left(\theta - \frac{\pi}{2}\right) &= \frac{1}{\cos\left(\theta - \frac{\pi}{2}\right)} && \text{using a reciprocal identity} \\
&= \frac{1}{\cos\left(-\left(\frac{\pi}{2} - \theta\right)\right)} && \text{factoring } -1 \text{ from } \theta - \frac{\pi}{2} \\
&= \frac{1}{\cos\left(\frac{\pi}{2} - \theta\right)} && \text{since } \cos(-x) = \cos(x) \\
&= \frac{1}{\sin(\theta)} && \text{since } \cos(x) = \sin\left(\frac{\pi}{2} - x\right)
\end{aligned}$$

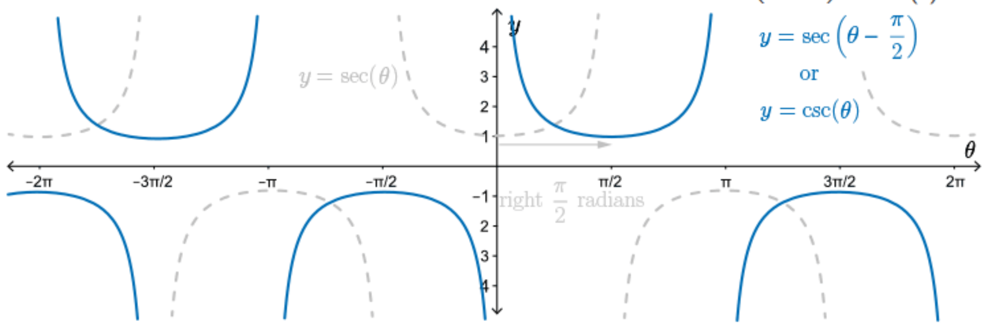
Examples

Example 3

Express $\sec\left(\theta - \frac{\pi}{2}\right)$ in terms of $\sin(\theta)$.

Solution

Notice that the graph of $y = \sec\left(\theta - \frac{\pi}{2}\right)$ can be obtained by translating the graph of $y = \sec(\theta)$ to the right by $\frac{\pi}{2}$ units. This results in the graph of $y = \csc(\theta)$, and thus it graphically verifies that $\sec\left(\theta - \frac{\pi}{2}\right) = \frac{1}{\sin(\theta)}$.



Due to the symmetry and periodic nature of trigonometric functions, equivalent trigonometric expressions can be identified or derived using transformations (particularly through reflections and horizontal translations).

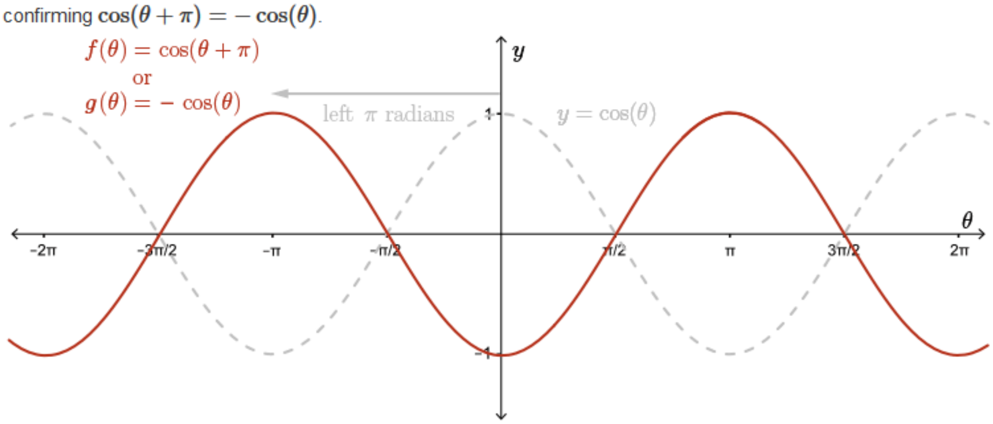
Examples

Example 4

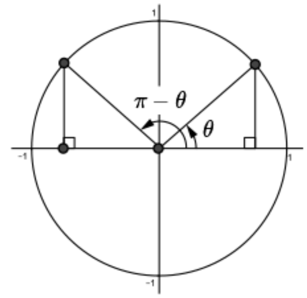
Show graphically that $\cos(\theta + \pi) = -\cos(\theta)$.

Solution

Consider $f(\theta) = \cos(\theta + \pi)$ and $g(\theta) = -\cos(\theta)$.
The graph of $y = f(\theta)$ can be obtained by translating the graph of $y = \cos(\theta)$ to the left π units. The graph of this function is the reflection of the cosine function in the x-axis and can therefore be defined by $g(\theta) = -\cos(\theta)$. Thus confirming $\cos(\theta + \pi) = -\cos(\theta)$.



Expanding on Example 4



We can see that this identity, $\cos(\theta + \pi) = -\cos(\theta)$, supports our understanding of the relationship between a principal angle and its reference or related acute angle.

For $0 < \theta < \frac{\pi}{2}$, an angle, $\pi + \theta$, in standard position, has a terminal arm in the third quadrant with a reference angle of θ.

Thus, $\cos(\pi + \theta) = -\cos(\theta)$.

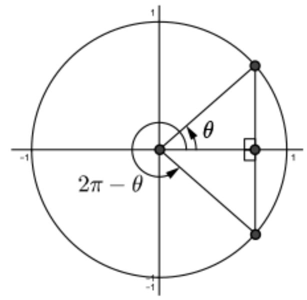
Similarly, by considering a principal angle in the other two quadrants, related to the reference angle, θ, we have

$$\cos(\pi - \theta) = -\cos(\theta)$$

and

$$\cos(2\pi - \theta) = \cos(\theta)$$

These statements are true for all values of θ. We can validate each statement using transformations.



Expanding on Example 4

Using the relationship between a principal angle and its reference or related acute angle, other identities using sine and tangent ratios can be identified. This list can be extended to include the reciprocal trigonometric ratios.

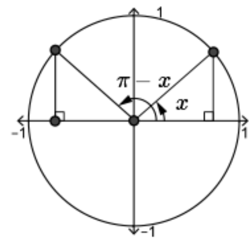
$\sin(\pi - \theta) = \sin(\theta)$	$\sin(\pi + \theta) = -\sin(\theta)$	$\sin(2\pi - \theta) = -\sin(\theta)$
$\cos(\pi - \theta) = -\cos(\theta)$	$\cos(\pi + \theta) = -\cos(\theta)$	$\cos(2\pi - \theta) = \cos(\theta)$
$\tan(\pi - \theta) = -\tan(\theta)$	$\tan(\pi + \theta) = \tan(\theta)$	$\tan(2\pi - \theta) = -\tan(\theta)$
$\csc(\pi - \theta) = \csc(\theta)$	$\csc(\pi + \theta) = -\csc(\theta)$	$\csc(2\pi - \theta) = -\csc(\theta)$
$\sec(\pi - \theta) = -\sec(\theta)$	$\sec(\pi + \theta) = -\sec(\theta)$	$\sec(2\pi - \theta) = -\sec(\theta)$
$\cot(\pi - \theta) = -\cot(\theta)$	$\cot(\pi + \theta) = \cot(\theta)$	$\cot(2\pi - \theta) = -\cot(\theta)$

Due to the periodic nature of trigonometric functions, these functions can be expressed in many equivalent forms.

Examples

Example 5

Express $\dfrac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{\pi}{2} - x\right)}$ in terms of $\sin(x)$.



Solution

$$\frac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{\pi}{2} - x\right)}$$

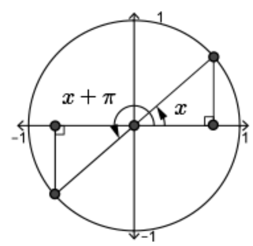
$$= \sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)\cos\left(\frac{\pi}{2} + x\right)\sin\left(\frac{3\pi}{2} - x\right)\tan\left(\frac{\pi}{2} - x\right)$$

$$= \sin(x)\cos(\pi + x)\tan(2\pi + x)\cos\left(\frac{\pi}{2} + x\right)\sin\left(\frac{3\pi}{2} - x\right)\tan\left(\frac{\pi}{2} - x\right)$$

Examples

Example 5

Express $\dfrac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{\pi}{2} - x\right)}$ in terms of $\sin(x)$.



Solution

$$\frac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{\pi}{2} - x\right)}$$

$$= \sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)\cos\left(\frac{\pi}{2} + x\right)\sin\left(\frac{3\pi}{2} - x\right)\tan\left(\frac{\pi}{2} - x\right)$$

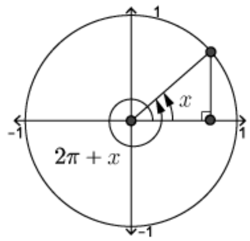
$$= \sin(x)\cos(\pi + x)\tan(2\pi + x)\cos\left(\frac{\pi}{2} + x\right)\sin\left(\frac{3\pi}{2} - x\right)\tan\left(\frac{\pi}{2} - x\right)$$

$$= \sin(x)(-\cos(x))\tan(2\pi + x)\cos\left(\frac{\pi}{2} + x\right)\sin\left(\frac{3\pi}{2} - x\right)\tan\left(\frac{\pi}{2} - x\right)$$

Examples

Example 5

Express $\dfrac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{\pi}{2} - x\right)}$ in terms of $\sin(x)$.



Solution

$$\frac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{\pi}{2} - x\right)}$$

$$= \sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)\cos\left(\frac{\pi}{2} + x\right)\sin\left(\frac{3\pi}{2} - x\right)\tan\left(\frac{\pi}{2} - x\right)$$

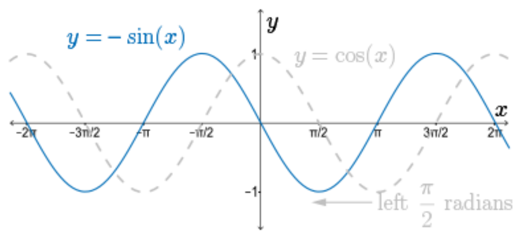
$$= \sin(x)(-\cos(x))\tan(2\pi + x)\cos\left(\frac{\pi}{2} + x\right)\sin\left(\frac{3\pi}{2} - x\right)\tan\left(\frac{\pi}{2} - x\right)$$

$$= \sin(x)(-\cos(x))\tan(x)\cos\left(\frac{\pi}{2} + x\right)\sin\left(\frac{3\pi}{2} - x\right)\tan\left(\frac{\pi}{2} - x\right)$$

Examples

Example 5

Express $\dfrac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{\pi}{2} - x\right)}$ in terms of $\sin(x)$.



Solution

$$\frac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{\pi}{2} - x\right)}$$

$$= \sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)\cos\left(\frac{\pi}{2} + x\right)\sin\left(\frac{3\pi}{2} - x\right)\tan\left(\frac{\pi}{2} - x\right)$$

$$= \sin(x)(-\cos(x))\tan(x)\cos\left(\frac{\pi}{2} + x\right)\sin\left(\frac{3\pi}{2} - x\right)\tan\left(\frac{\pi}{2} - x\right)$$

$$= \sin(x)(-\cos(x))\tan(x)(-\sin x)\sin\left(\frac{3\pi}{2} - x\right)\tan\left(\frac{\pi}{2} - x\right)$$

Examples

Example 5

Express $\dfrac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{\pi}{2} - x\right)}$ in terms of $\sin(x)$.

Solution

$$\frac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{\pi}{2} - x\right)}$$

$$= \sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)\cos\left(\frac{\pi}{2} + x\right)\sin\left(\frac{3\pi}{2} - x\right)\tan\left(\frac{\pi}{2} - x\right)$$

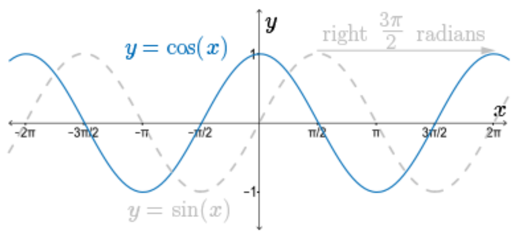
$$= \sin(x)(-\cos(x))\tan(x)(-\sin x)\sin\left(\frac{3\pi}{2} - x\right)\tan\left(\frac{\pi}{2} - x\right)$$

$$= \sin(x)(-\cos(x))\tan(x)(-\sin x)\sin\left(-\left(x - \frac{3\pi}{2}\right)\right)\tan\left(\frac{\pi}{2} - x\right)$$

Examples

Example 5

Express $\dfrac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{\pi}{2} - x\right)}$ in terms of $\sin(x)$.



Solution

$$\frac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{\pi}{2} - x\right)}$$

$$= \sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)\cos\left(\frac{\pi}{2} + x\right)\sin\left(\frac{3\pi}{2} - x\right)\tan\left(\frac{\pi}{2} - x\right)$$

$$= \sin(x)(-\cos(x))\tan(x)(-\sin(x))\left(-\sin\left(x - \frac{3\pi}{2}\right)\right)\tan\left(\frac{\pi}{2} - x\right)$$

$$= \sin(x)(-\cos(x))\tan(x)(-\sin(x))(-\cos(x))\tan\left(\frac{\pi}{2} - x\right)$$

Examples

Example 5

Express $\dfrac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{\pi}{2} - x\right)}$ in terms of $\sin(x)$.

Solution

$$\frac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{\pi}{2} - x\right)}$$

$$= \sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)\cos\left(\frac{\pi}{2} + x\right)\sin\left(\frac{3\pi}{2} - x\right)\tan\left(\frac{\pi}{2} - x\right)$$

$$= \sin(x)(-\cos(x))\tan(x)(-\sin(x))(-\cos(x))\tan\left(\frac{\pi}{2} - x\right)$$

$$= \sin(x)(-\cos(x))\tan(x)(-\sin(x))(-\cos(x))(\cot(x))$$

Examples

Example 5

Express $\dfrac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{\pi}{2} - x\right)}$ in terms of $\sin(x)$.

Solution

$$\frac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{\pi}{2} - x\right)}$$

$$= \sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)\cos\left(\frac{\pi}{2} + x\right)\sin\left(\frac{3\pi}{2} - x\right)\tan\left(\frac{\pi}{2} - x\right)$$

$$= \sin(x)(-\cos(x))\tan(x)(-\sin(x))(-\cos(x))(\cot(x))$$

$$= -\sin^2(x)\cos^2(x)\left(\frac{\sin(x)}{\cos(x)}\right)\left(\frac{\cos(x)}{\sin(x)}\right)$$

$$= -\sin^2(x)\cos^2(x)$$

$$= -\sin^2(x)(1 - \sin^2(x))$$

$$= \sin^4(x) - \sin^2(x)$$

Therefore, $\dfrac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi + x)}{\sec\left(\frac{\pi}{2} + x\right)\csc\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{\pi}{2} - x\right)} = \sin^4(x) - \sin^2(x),\ x \neq \frac{n}{2}\pi,\ n \in \mathbb{Z}$.

Summary

The ability to simplify trigonometric expressions or identify equivalent forms allows for more efficient problem solving when studying trigonometric relations.

In this module, we have identified many trigonometric identities, some by their definition and others by relationships that exist in the ratios or the graphs of the functions. Some fundamental identities include:

Reciprocal Identities

$$\csc(\theta)=\frac{1}{\sin(\theta)} \qquad \sec(\theta)=\frac{1}{\cos(\theta)} \qquad \cot(\theta)=\frac{1}{\tan(\theta)}$$

Quotient Identities

$$\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)} \qquad \cot(\theta)=\frac{\cos(\theta)}{\sin(\theta)}$$

Pythagorean Identities

$$\sin^2(\theta)+\cos^2(\theta)=1 \qquad 1+\cot^2(\theta)=\csc^2(\theta) \qquad \tan^2(\theta)+1=\sec^2(\theta)$$

Cofunction Identities

$$\sin(\theta)=\cos\left(\frac{\pi}{2}-\theta\right) \qquad \cos(\theta)=\sin\left(\frac{\pi}{2}-\theta\right) \qquad \tan(\theta)=\cot\left(\frac{\pi}{2}-\theta\right)$$

Symmetry

$$\sin(-\theta)=-\sin(\theta) \qquad \cos(-\theta)=\cos(\theta) \qquad \tan(-\theta)=-\tan(\theta)$$

Translations and Reflections

$$\sin(\pi-\theta)=\sin(\theta) \qquad \sin(\pi+\theta)=-\sin(\theta) \qquad \sin(2\pi-\theta)=-\sin(\theta)$$

$$\cos(\pi-\theta)=-\cos(\theta) \qquad \cos(\pi+\theta)=-\cos(\theta) \qquad \cos(2\pi-\theta)=\cos(\theta)$$

$$\tan(\pi-\theta)=-\tan(\theta) \qquad \tan(\pi+\theta)=\tan(\theta) \qquad \tan(2\pi-\theta)=-\tan(\theta)$$