The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Solving Trigonometric Equations

In This Module

- We will discuss a variety of strategies to solve both linear and nonlinear trigonometric equations algebraically.
- We will use fundamental identities and formulas studied in this unit to assist us in solving equations involving more than one trigonometric function.
- We will discuss how graphing technology can be used to solve trigonometric equations and verify solutions.

Examples

Example 1

Solve $\csc(x) + 6 = 1 - 3\csc(x)$, where $0 \leq x \leq 2\pi$, correct to two decimal places. Verify the solution using graphing technology.

Solution

First, we identify any non-permissible values of x.
Cosecant of x is undefined when $x = n\pi, n \in \mathbb{Z}$.
Thus, $x \neq n\pi, n \in \mathbb{Z}$.
To solve this equation, we can use the same techniques used in solving linear equations.
By rearranging the terms and simplifying, we isolate the trigonometric ratio, $\csc(x)$.

$$\begin{aligned} \csc(x) + 6 &= 1 - 3\csc(x) \\ 4\csc(x) &= -5 \\ \csc(x) &= -\frac{5}{4} \\ \sin(x) &= -\frac{4}{5} \\ x &= \sin^{-1}\left(-\frac{4}{5}\right) \\ x &\approx -0.9273 \end{aligned}$$

This answer provides us with the related acute angle, or reference angle, which is approximately 0.9273 radians.

Examples

Example 1

Solve $\csc(x) + 6 = 1 - 3\csc(x)$, where $0 \leq x \leq 2\pi$, correct to two decimal places. Verify the solution using graphing technology.

Solution

The sine function is negative for any angle with a terminal arm in quadrant 3 or quadrant 4.

$$x \approx \pi + 0.9273 \text{ or } x \approx 2\pi - 0.9273$$

To two decimal places of accuracy,

$$x \approx 4.07 \text{ radians or } x \approx 5.36 \text{ radians}$$

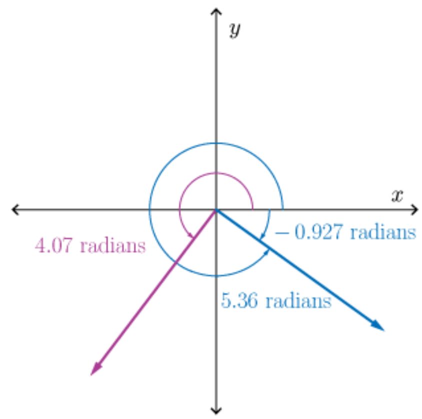
All possible solutions are defined by

$$x \approx 4.0689 + 2\pi n, n \in \mathbb{Z} \text{ or } x \approx 5.3559 + 2\pi n, n \in \mathbb{Z}$$

However, 4.07 and 5.36 are the only two solutions within the specified domain from $0 \leq x \leq 2\pi$.

Note that these values are permissible values of the cosecant function.

Therefore, the roots in the domain $0 \leq x \leq 2\pi$ are approximately 4.07 and 5.36.



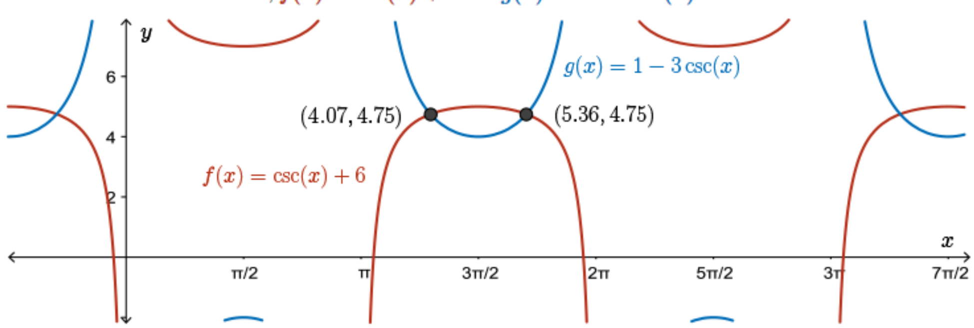
Examples

Example 1

Solve $\csc(x) + 6 = 1 - 3\csc(x)$, where $0 \leq x \leq 2\pi$, correct to two decimal places. Verify the solution using graphing technology.

Solution

Therefore, the roots in the domain $0 \leq x \leq 2\pi$ are approximately 4.07 and 5.36.
We will now verify the solutions to this equation using graphing technology. One way to do this is to determine the point of intersection of the two functions, $f(x) = \csc(x) + 6$ and $g(x) = 1 - 3\csc(x)$.



Using graphing technology, we see that $f(x) = g(x)$ at $x \approx 4.07$ and $x \approx 5.36$, thus verifying the solution. Note that this same approach can be used to solve trigonometric equations using graphing technology.

Examples

Example 2

Determine the exact roots of $\sqrt{2}\sin(x)\cos(x) = \cos(x)$, where $-\pi \le x \le 2\pi$.

Solution

First, we note that there are no restrictions on the variable x, since sine and cosine are defined for all real values of x.

$$\frac{\sqrt{2}\sin(x)\cos(x)}{\cos(x)} = \frac{\cos(x)}{\cos(x)}$$

$$\cancel{\frac{\sqrt{2}\sin(x)\cos(x)}{\cos(x)}} = \cancel{\frac{\cos(x)}{\cos(x)}}$$

Examples

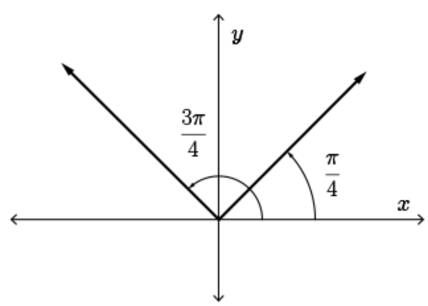
Example 2

Determine the exact roots of $\sqrt{2}\sin(x)\cos(x) = \cos(x)$, where $-\pi \le x \le 2\pi$.

Solution

Collect the terms to one side of the equation and factor,

$$\begin{aligned}\sqrt{2}\sin(x)\cos(x) &= \cos(x)\\ \sqrt{2}\sin(x)\cos(x) - \cos(x) &= 0\\ \cos(x)\left(\sqrt{2}\sin(x) - 1\right) &= 0\\ \cos(x) = 0 \text{ or } \sqrt{2}\sin(x) - 1 &= 0\end{aligned}$$



We know $\cos(x) = 0$ when $x = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}$.
Thus, for $-\pi \le x \le 2\pi$, $x = -\frac{\pi}{2}$, $\frac{\pi}{2}$, and $\frac{3\pi}{2}$.

We know $\sqrt{2}\sin(x) - 1 = 0$ when $\sin(x) = \frac{1}{\sqrt{2}}$.
The reference angle is $\frac{\pi}{4}$ and sine is positive for any angle with a terminal arm in quadrant 1 or quadrant 2.
For $0 \le x \le 2\pi$, $x = \frac{\pi}{4}$ and $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$.
For $-\pi \le x \le 0$, $\sin(x) \le 0$ so no additional solutions lie in this part of the required domain.
Thus, $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.

Therefore, the exact roots of $\sqrt{2}\sin(x)\cos(x) = \cos(x)$, where $-\pi \le x \le 2\pi$ are $-\frac{\pi}{2}$, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, and $\frac{3\pi}{2}$.

Examples

Example 2

Determine the exact roots of $\sqrt{2}\sin(x)\cos(x) = \cos(x)$, where $-\pi \le x \le 2\pi$.

Solution

One way to verify this solution graphically using technology, is to graph

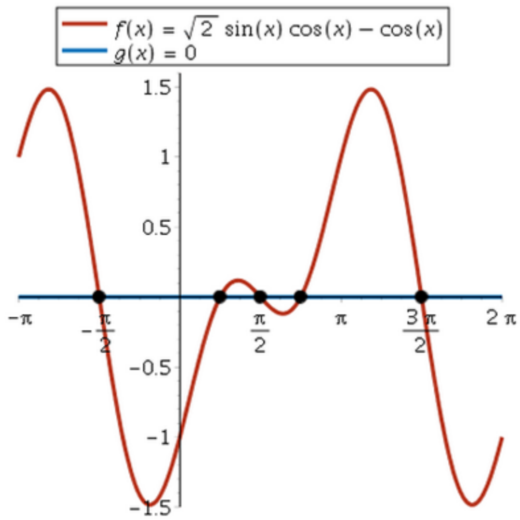
$$f(x) = \sqrt{2}\sin(x)\cos(x) - \cos(x)$$

and determine the zeros or x-intercepts of the function.

The graph presented here allows for a quick visual check of the solution.

The zeros occur occur when $f(x) = 0$; that is, when $\sqrt{2}\sin(x)\cos(x) - \cos(x) = 0$.

Zeros shown in the domain $-\pi \le x \le 2\pi$ are $-\frac{\pi}{2}$, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, and $\frac{3\pi}{2}$.



Examples

Example 3

Solve $3\tan^2(x) + \tan(x) = 2 + 2\tan(x)$, where $0^\circ \le x \le 360^\circ$. Round answers to one decimal place of accuracy when necessary.

Solution

$$3\tan^2(x) + \tan(x) = 2 + 2\tan(x),\ x \ne 90^\circ + 180^\circ n,\ n \in \mathbb{Z}$$

We rearrange the equation obtaining

$$3\tan^2(x) - \tan(x) - 2 = 0$$

To assist in factoring, we may wish to make a substitution of $t = \tan(x)$:

$$\begin{aligned} 3(\tan(x))^2 - \tan(x) - 2 &= 0 \\ 3t^2 - t - 2 &= 0 \\ (3t+2)(t-1) &= 0 \\ t = -\frac{2}{3} &\text{ or } t = 1 \end{aligned}$$

Thus, $\tan(x) = -\frac{2}{3}$ or $\tan(x) = 1$.

Alternatively,

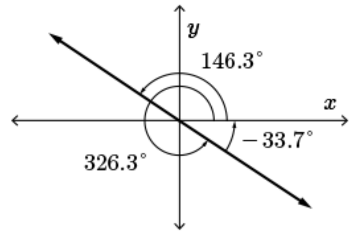
$$\begin{aligned} 3(\tan(x))^2 - \tan(x) - 2 &= 0 \\ \left(3\tan(x) + 2\right)\left(\tan(x) - 1\right) &= 0 \\ \tan(x) = -\frac{2}{3} &\text{ or } \tan(x) = 1 \end{aligned}$$

Examples

Example 3

Solve $3\tan^2(x)+\tan(x)=2+2\tan(x)$, where $0^\circ \le x \le 360^\circ$. Round answers to one decimal place of accuracy when necessary.

Solution



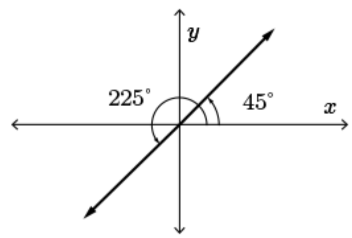
We must now solve $\tan(x)=-\dfrac{2}{3}$ and $\tan(x)=1$.

For $\tan(x)=-\dfrac{2}{3}$, $x=\tan^{-1}\left(-\dfrac{2}{3}\right)$. Thus,

$$x \approx -33.7^\circ$$

The reference angle is approximately 33.7°.
The tangent function is negative for any angle with a terminal arm in quadrant 2 or quadrant 4, so

$$x \approx 180^\circ - 33.7^\circ = 146.3^\circ \text{ or } x \approx 360^\circ - 33.7^\circ = 326.3^\circ$$



For $\tan(x)=1$, we note the tangent function is positive for any angle with a terminal arm in quadrant 1 or quadrant 3.
Thus, $x=45^\circ$ or $x=180^\circ+45^\circ=225^\circ$.
These solutions can be obtained using the 1, 1, $\sqrt{2}$ triangle for 45°.

Therefore, the solutions are 45°, 146.3°, 225°, and 326.3°.
Remember to check that non-permissible values of x are not included in the solution.

Examples

Example 4

Solve $\cos^2(x)=3-4\sin(x)$ in the interval $[0,3\pi]$. Round solutions to two decimal places of accuracy.

Solution

Using a Pythagorean identity, we can replace $\cos^2(x)$ with $1-\sin^2(x)$ to obtain a quadratic equation in terms of $\sin(x)$.

$$\begin{aligned}\cos^2(x) &= 3-4\sin(x)\\ 1-\sin^2(x) &= 3-4\sin(x)\\ \sin^2(x)-4\sin(x)+2 &= 0\end{aligned}$$

The quadratic formula will provide the values for $\sin(x)$.
Substituting $a=1$, $b=-4$, and $c=2$, we have

$$\begin{aligned}\sin(x) &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}\\ \sin(x) &= \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(2)}}{2(1)}\\ \sin(x) &= \frac{4\pm2\sqrt{2}}{2}\\ \sin(x) &= 2\pm\sqrt{2}\end{aligned}$$

Examples

Example 4

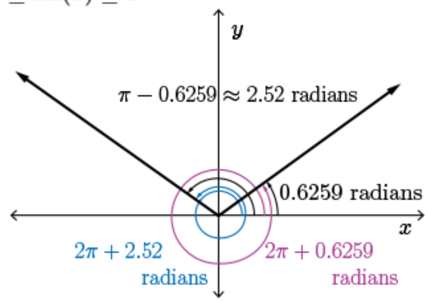
Solve $\cos^2(x) = 3 - 4\sin(x)$ in the interval $[0, 3\pi]$. Round solutions to two decimal places of accuracy.

Solution

We must now solve $\sin(x) = 2 + \sqrt{2}$ and $\sin(x) = 2 - \sqrt{2}$.

No solution exists for $\sin(x) = 2 + \sqrt{2}$, since $2 + \sqrt{2} > 1$ and $-1 \leq \sin(x) \leq 1$.

$$\begin{aligned} \sin(x) &= 2 - \sqrt{2} \\ x &= \sin^{-1}(2 - \sqrt{2}) \\ x = 0.6259 \ \text{ or } x &= \pi - 0.6259 \end{aligned}$$



Therefore, for $x \in [0, 2\pi]$ we have $x \approx 0.63$ or $x \approx 2.52$.
All possible solutions include $0.6259 + 2\pi n$ and $2.5157 + 2\pi n$, $n \in \mathbb{Z}$.
So, $0.6259 + 2\pi$ and $2.5157 + 2\pi$ are also roots to the equation in the interval from 2π to 3π.

Therefore, the roots to the equation in the interval $[0, 3\pi]$ are approximately 0.63, 2.52, 6.91, and 8.80.

Examples

Example 5

Determine the roots of $\cos(x) - \sin(x) = \cos(2x)$ where $-\pi \leq x \leq \pi$.

Solution

Replacing $\cos(2x)$ with $\cos^2(x) - \sin^2(x)$, we have

$$\begin{aligned} \cos(x) - \sin(x) &= \cos(2x) \\ \cos(x) - \sin(x) &= \cos^2(x) - \sin^2(x) \\ \cos(x) - \sin(x) &= \big(\cos(x) + \sin(x)\big)\big(\cos(x) - \sin(x)\big) \\ 0 &= \big(\cos(x) + \sin(x)\big)\big(\cos(x) - \sin(x)\big) - \big(\cos(x) - \sin(x)\big) \\ 0 &= \big(\cos(x) - \sin(x)\big)\big(\cos(x) + \sin(x) - 1\big) \end{aligned}$$

So, $\cos(x) - \sin(x) = 0$ or $\cos(x) + \sin(x) - 1 = 0$.

Examples

Example 5

Determine the roots of $\cos(x) - \sin(x) = \cos(2x)$ where $-\pi \le x \le \pi$.

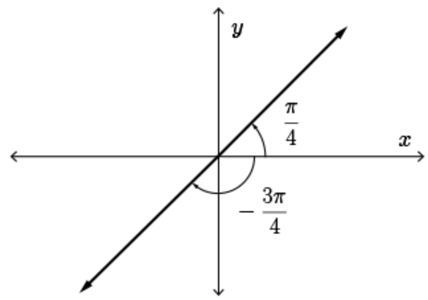
Solution

Solving the first equation:

$$\cos(x) - \sin(x) = 0$$
$$\sin(x) = \cos(x)$$

Dividing each side by $\cos(x)$, we obtain

$$\frac{\sin(x)}{\cos(x)} = \frac{\cos(x)}{\cos(x)}$$
$$\tan(x) = 1$$



The reference angle of the solution is $\frac{\pi}{4}$.

Tangent is positive for any angle with a terminal arm in quadrant 1 or quadrant 3.

For $-\pi \le x \le \pi$, $\tan(x) = 1$ when $x = \frac{\pi}{4}$ or $x = -\frac{3\pi}{4}$.

Examples

Example 5

Determine the roots of $\cos(x) - \sin(x) = \cos(2x)$ where $-\pi \le x \le \pi$.

Solution

Solving the second equation, $\cos(x) + \sin(x) - 1 = 0$:

$$\cos(x) + \sin(x) = 1$$
$$\Big(\cos(x) + \sin(x)\Big)^2 = 1^2$$
$$\cos^2(x) + 2\sin(x)\cos(x) + \sin^2(x) = 1$$
$$2\sin(x)\cos(x) + 1 = 1$$
$$2\sin(x)\cos(x) = 0$$

Replacing $2\sin(x)\cos(x)$ with $\sin(2x)$, we have

$$\sin(2x) = 0$$

To determine the values of x in the domain, $-\pi \le x \le \pi$, find values for $2x$ where $-2\pi \le 2x \le 2\pi$.

We know $\sin(\theta) = 0$ when $\theta = n\pi$, $n \in \mathbb{Z}$ so

$$2x = -2\pi, -\pi, 0, \pi, 2\pi$$
$$x = -\pi, -\tfrac{\pi}{2}, 0, \tfrac{\pi}{2}, \pi$$

Alternatively,

$$2\sin(x)\cos(x) = 0$$

when $\sin(x) = 0$ or $\cos(x) = 0$.

Thus, $x = -\pi, 0, \pi$ or $x = -\frac{\pi}{2}, \frac{\pi}{2}$.

Examples

Example 5

Determine the roots of $\cos(x) - \sin(x) = \cos(2x)$ where $-\pi \leq x \leq \pi$.

Solution

Since squaring was used to solve the equation $\cos(x) + \sin(x) - 1 = 0$, we must check for extraneous roots (roots that are not solutions to the original equation).
Which of these values for x, $-\pi$, $-\frac{\pi}{2}$, 0, $\frac{\pi}{2}$, and π, are actual solutions to $\cos(x) + \sin(x) - 1 = 0$?

We can show that

$$\cos(-\pi) + \sin(-\pi) - 1 \neq 0, \cos\left(-\frac{\pi}{2}\right) + \sin\left(-\frac{\pi}{2}\right) - 1 \neq 0, \text{ and } \cos(\pi) + \sin(\pi) - 1 \neq 0.$$

However, $x = 0$ and $x = \frac{\pi}{2}$ satisfy $\cos(x) + \sin(x) - 1 = 0$, so 0 and $\frac{\pi}{2}$ are the roots of this equation in the specified domain.

Examples

Example 5

Determine the roots of $\cos(x) - \sin(x) = \cos(2x)$ where $-\pi \leq x \leq \pi$.

Solution

Summarizing,

$$\begin{aligned} \cos(x) - \sin(x) &= \cos(2x) \\ \cos(x) - \sin(x) &= \cos^2(x) - \sin^2(x) \\ 0 &= \Big(\cos(x) + \sin(x)\Big)\Big(\cos(x) + \sin(x) - 1\Big) \end{aligned}$$

Thus, either $\cos(x) - \sin(x) = 0$ or $\cos(x) + \sin(x) - 1 = 0$.
We have determined that $\cos(x) - \sin(x) = 0$ has solutions $-\frac{3\pi}{4}$ and $\frac{\pi}{4}$ and $\cos(x) + \sin(x) - 1 = 0$ has solutions 0 and $\frac{\pi}{2}$. So $x = 0$ and $x = \frac{\pi}{2}$ are the roots of this equation in the specified domain.
Therefore, the roots of the equation, $\cos(x) - \sin(x) = \cos(2x)$, in the domain $-\pi \leq x \leq \pi$ are $-\frac{3\pi}{4}$, 0, $\frac{\pi}{4}$, and $\frac{\pi}{2}$.

Examples

Example 5

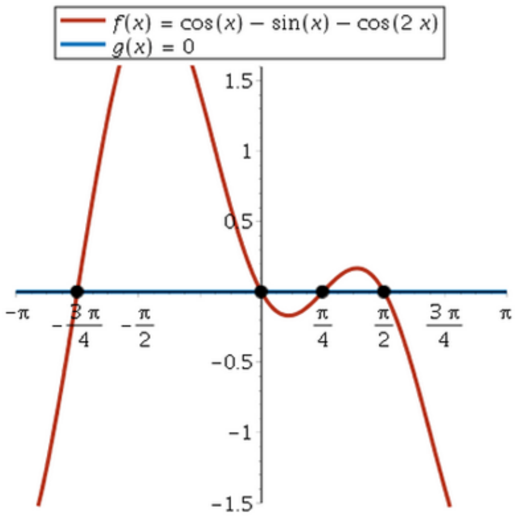
Determine the roots of $\cos(x) - \sin(x) = \cos(2x)$ where $-\pi \leq x \leq \pi$.

Solution

The graph shown here verifies the solutions for $-\pi \leq x \leq \pi$.

The zeros of the function, $f(x) = \cos(x) - \sin(x) - \cos(2x)$, are the roots of the equation

$$\cos(x) - \sin(x) = \cos(2x)$$



In Summary

- Linear trigonometric equations involving one trigonometric function can be solved algebraically using the same techniques used to solve linear equations.
- Solve second-degree equations, involving one trigonometric function, by factoring or using the quadratic formula.
- If the equation involves more than one trigonometric function, try separating the functions by factoring, or make substitutions using fundamental identities to express the equation in terms of one trigonometric function.
- Keep answers exact, using the special triangles for acute angles $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$, whenever possible.
- Due to the periodic nature of trigonometric functions, trigonometric equations may have an infinite number of solutions. For this reason, a defined domain is often specified. Be sure to find all roots to the equation within the indicated domain. Solutions must be stated using the same unit of measure as the domain.
- Keep in mind any restrictions on the variable of the equation due to the trigonometric functions involved. Non-permissible values should never be included in the solution.
- Trigonometric equations can be solved or solutions verified graphically using technology.