The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Optimization

In This Module

- We revisit one of the most important applications of differential calculus: optimization.
- The problems we consider will involve exponential and trigonometric modelling functions.

Examples

Example 1

The net monthly profit, in dollars, from the sale of a certain item is given by the formula $P(x) = 60x^2e^{-0.25x} + 40$, where $x \geq 1$ is the number of items sold, in hundreds.

a. If the factory can produce up to 1000 items per month, determine the number of items that yield the maximum profit. What is this maximum profit?

Solution

Since $x = 10$ corresponds to $10 \times 100 = 1000$ units of the item, we need to determine the maximum value of the profit function $P(x)$ for $1 \leq x \leq 10$. Since $P(x)$ is continuous over all of $\mathbb{R}$, it is continuous over the interval $[1, 10]$.
By the extreme value theorem, $P(x)$ must attain a maximum value on the interval $[1, 10]$.
Let's locate the critical points of $P(x)$ over this interval.
Using the product rule and the chain rule, we differentiate $P(x)$:

$$\begin{aligned} P'(x) &= 60\left[x^2(-0.25)\left(e^{-0.25x}\right) + e^{-0.25x}(2x)\right] \\ &= -15x^2e^{-0.25x} + 120xe^{-0.25x} \end{aligned}$$

We have

$$\begin{aligned} P'(x) &= 0 \\ -15x^2e^{-0.25x} + 120xe^{-0.25x} &= 0 \\ -x^2e^{-0.25x} + 8xe^{-0.25x} &= 0 \\ e^{-0.25x}x(8-x) &= 0 \end{aligned}$$

Examples

Example 1

The net monthly profit, in dollars, from the sale of a certain item is given by the formula $P(x) = 60x^2e^{-0.25x} + 40$, where $x \geq 1$ is the number of items sold, in hundreds.
a. If the factory can produce up to 1000 items per month, determine the number of items that yield the maximum profit. What is this maximum profit?

Solution

$$e^{-0.25x}x(8-x) = 0$$

Since $e^{-0.25x} \neq 0$ for all x, we have $P'(x) = 0$ if and only if $x = 0$ or $x = 8$.
Therefore, $P(x)$ has two critical points, but only one, $x = 8$, lies in the interval $[1, 10]$.
We evaluate the function at the critical point and the endpoints of the interval to locate the maximum value of $P(x)$:

$$P(1) = 60(1)^2e^{-0.25(1)} + 40 = 60e^{-0.25} + 40 \approx 86.73$$
$$P(8) = 60(8)^2e^{-0.25(8)} + 40 = 3840e^{-2} + 40 \approx 559.69$$
$$P(10) = 60(10)^2e^{-0.25(10)} + 40 = 6000e^{-2.5} + 40 \approx 532.51$$

Therefore, $P(x)$ attains its maximum value at $x = 8$.
We conclude that the maximum profit attainable is $\$559.69$, which is obtained by selling 800 items.

Examples

Example 1

The net monthly profit, in dollars, from the sale of a certain item is given by the formula $P(x) = 60x^2e^{-0.25x} + 40$, where $x \geq 1$ is the number of items sold, in hundreds.
b. Repeat part a) assuming that the factory can produce only 500 items per month at full capacity.

Solution

Now we need to determine the maximum value of the profit function, $P(x)$, for $1 \leq x \leq 5$.
From part a), we know that $P(x)$ has no critical points on this interval and so the maximum value must occur at one of the endpoints.
We have $P(1) = 86.73$ and $P(5) \approx 469.76$ and so $P(x)$ attains its maximal value at $x = 5$, corresponding to 500 items.
The maximum profit is $\$469.76$.

Examples

Example 2

A student's success in a course depends on how many hours the student studies.
Suppose that Jenny is studying for an exam and because of the nature of the course, the effectiveness of studying can be measured using the formula $E(t) = 0.6(8 + te^{-\frac{t}{15}})$ where t is the number of hours spent studying.
If Jenny has up to 20 hours for studying, how many hours should be spent to maximize effectiveness?

Kirkpatrick, C., & Crippin, P. (2009). Calculus and Vectors (p. 241). Toronto, ON: Nelson Education.

Solution

We need to determine the maximum value of the function $E(t)$ on the interval $0 \leq t \leq 20$.
First, we differentiate the function, with respect to time, in order to locate the critical points:

$$\begin{aligned} E'(t) &= 0.6\left[t\left(-\frac{1}{15}e^{-\frac{t}{15}}\right) + e^{-\frac{t}{15}}(1)\right] \\ &= -0.04te^{-\frac{t}{15}} + 0.6e^{-\frac{t}{15}} \\ &= e^{-\frac{t}{15}}(0.6 - 0.04t) \end{aligned}$$

Since $e^{-\frac{t}{15}} \neq 0$ for all t, then $E'(t) = 0$ exactly when $t = \frac{0.6}{0.04} = 15$, so $E(t)$ has one critical point in the interval $[0, 20]$.

Examples

Example 2

A student's success in a course depends on how many hours the student studies.
Suppose that Jenny is studying for an exam and because of the nature of the course, the effectiveness of studying can be measured using the formula $E(t) = 0.6(8 + te^{-\frac{t}{15}})$ where t is the number of hours spent studying.
If Jenny has up to 20 hours for studying, how many hours should be spent to maximize effectiveness?

Kirkpatrick, C., & Crippin, P. (2009). Calculus and Vectors (p. 241). Toronto, ON: Nelson Education.

Solution

We now evaluate the function at the critical point, $t = 15$, and the endpoints of the interval to locate the maximum value of $E(t)$:

$$\begin{aligned} E(0) &= 0.6(8 + (0)e^{-\frac{0}{15}}) = 0.6(8) = 4.80 \\ E(15) &= 0.6(8 + 15e^{-\frac{15}{15}}) = 0.6(8 + 15e^{-1}) \approx 8.11 \\ E(20) &= 0.6(8 + 20e^{-\frac{20}{15}}) = 0.6(8 + 20e^{-\frac{4}{3}}) \approx 7.96 \end{aligned}$$

Therefore, the maximum value occurs at $t = 15$.
To obtain the maximum effectiveness of 8.11, Jenny should study for 15 hours.

Examples

Example 3

The vertical displacement of the body of a car after driving over a speed bump is modelled by the function $h(t) = e^{-0.4t}\sin(t)$ where h is measured in meters and $t \geq 0$ is measured in seconds.
At $t = 0$, just as the car is about to pass over the speed bump, we have $h(0) = 0$, which corresponds to the resting height of the body of the car.
If $h(t) > 0$, then the car's body is higher than its resting height at time t, and if $h(t) < 0$ then it is lower than its resting height.

Determine when the maximum vertical displacement occurs and find this maximum displacement.

Solution

To find the maximum vertical displacement, we need to locate the times when the car's body is farthest from its resting height, either in the positive or the negative direction.
Therefore, we need to find the absolute maximum and minimum values attained by the function $h(t)$ over the interval $t \geq 0$, and determine which value is largest in magnitude.
To do so, we first need to locate the critical points of the function $h(t)$.
Using the product rule and the chain rule, we differentiate $h(t)$ and find:

$$\begin{aligned} h'(t) &= e^{-0.4t}\frac{d}{dt}\Big(\sin(t)\Big) + \sin(t)\frac{d}{dt}\Big(e^{-0.4t}\Big) \\ &= e^{-0.4t}\cos(t) + \sin(t)\left(-0.4e^{-0.4t}\right) \\ &= e^{-0.4t}\left(\cos(t) - 0.4\sin(t)\right) \end{aligned}$$

Examples

Example 3

The vertical displacement of the body of a car after driving over a speed bump is modelled by the function $h(t) = e^{-0.4t}\sin(t)$ where h is measured in meters and $t \geq 0$ is measured in seconds.
Determine when the maximum vertical displacement occurs, and find this maximum displacement.

Solution

$$h'(t) = e^{-0.4t}\left(\cos(t) - 0.4\sin(t)\right)$$

Since $h'(t)$ is defined for all t, the only critical points occur when $h'(t) = 0$. Since $e^{-0.4t} \neq 0$ for all t, we have $h'(t) = 0$ exactly when

$$\begin{aligned} \cos(t) - 0.4\sin(t) &= 0 \\ \cos(t) &= 0.4\sin(t) \\ \frac{\sin(t)}{\cos(t)} &= \frac{1}{0.4} \\ \tan(t) &= 2.5 \end{aligned}$$

Therefore, $t = \tan^{-1}(2.5) \approx 1.19$, and since $\tan(t)$ has period π, we have $h'(t) = 0$ when $t \approx 1.19 + k\pi$ for any integer k.
Therefore, the function $h(t)$ has infinitely many critical points, each corresponding to a local extreme.

Examples

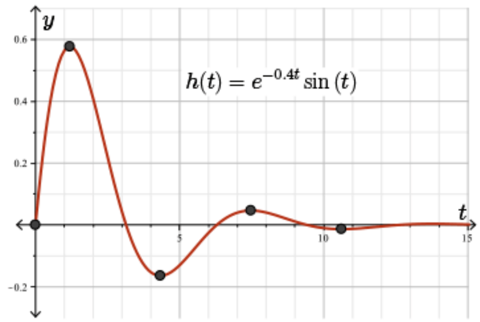
Example 3

The vertical displacement of the body of a car after driving over a speed bump is modelled by the function $h(t) = e^{-0.4t}\sin(t)$ where h is measured in meters and $t \geq 0$ is measured in seconds.
Determine when the maximum vertical displacement occurs, and find this maximum displacement.

Solution

Plotting the function values at a few of the critical points (and using the algorithm from a previous module), we can make the following sketch of $h(t)$:

$t \approx$	1.19	4.33	7.47	10.61
$h(t) \approx$	0.58	-0.16	0.05	-0.01



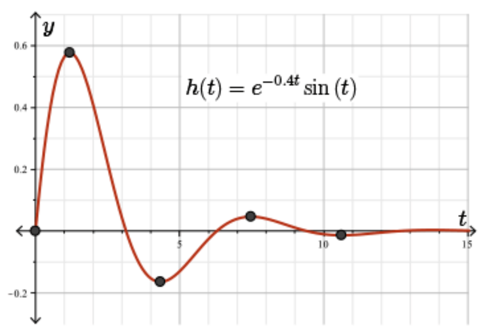
Examples

Example 3

The vertical displacement of the body of a car after driving over a speed bump is modelled by the function $h(t) = e^{-0.4t}\sin(t)$ where h is measured in meters and $t \geq 0$ is measured in seconds.
Determine when the maximum vertical displacement occurs, and find this maximum displacement.

Solution

$t \approx$	1.19	4.33	7.47	10.61
$h(t) \approx$	0.58	-0.16	0.05	-0.01



Since the function $h(t)$ involves $\sin(t)$, the curve oscillates about the t-axis as we would expect.
As the term $\sin(t)$ is multiplied by the decreasing function $e^{-0.4t}$, we see that the amplitude of the oscillations decreases as t increases.
Therefore, the critical point $t \approx 1.19$ corresponds to the absolute maximum of the function $h(t)$ and the critical point $t \approx 4.33$ corresponds to the absolute minimum.
Clearly, the maximum displacement occurs in the positive direction at $t \approx 1.19$.

Therefore, the maximum displacement occurs approximately 1.19 seconds after the car hits the bump, and the maximum displacement is approximately 0.58 m (upwards).

Examples

Example 3

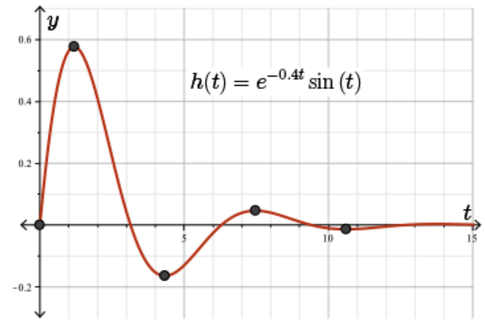
The vertical displacement of the body of a car after driving over a speed bump is modelled by the function $h(t) = e^{-0.4t}\sin(t)$ where h is measured in meters and $t \geq 0$ is measured in seconds.
Determine when the maximum vertical displacement occurs, and find this maximum displacement.

Solution

What is the physical meaning of the absolute minimum value of $h(4.33) \approx -0.16$ m?

Recall that $h = 0$ corresponds to the resting height of the car's body, not a height of 0 as measured from the ground.

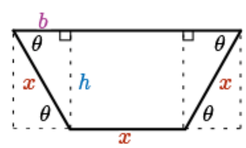
Since the body of the car moves -0.16 m downward, if the bottom of the car's body rests less than 0.16 m above the ground, then this speed bump will cause the body of the car to crash into the ground.



Examples

Example 4

A company makes troughs for holding water for farm animals.
The frame of the trough is made by taking a piece of sheet metal and folding $\frac{1}{3}$ of the sheet up on each side, so that each side makes an angle θ with the ground.
Assuming that the original piece of sheet metal is 45 cm wide, determine the angle θ that maximizes the volume of water that the trough can hold.
What is the cross-sectional area of this maximal trough?



Solution

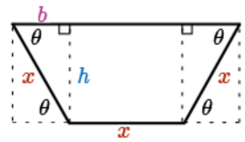
First, we note that we only have a proper trough if $0 < \theta \leq \frac{\pi}{2}$. Why?
We need to keep this physical restriction in mind while we are solving the problem.

To maximize the volume of water in the trough, we need to maximize the trapezoidal cross-sectional area of the trough.
The area of the trapezoid can be divided into the area of a rectangle and the areas of 2 congruent right-angled triangles as shown.
We denote the base and height of these two triangles by b and h respectively.
Using geometry, we know that the angle between the base b and the hypotenuse x must also be equal to θ.

Examples

Example 4

A company makes troughs for holding water for farm animals.
The frame of the trough is made by taking a piece of sheet metal and folding $\frac{1}{3}$ of the sheet up on each side, so that each side makes an angle θ with the ground.
Assuming that the original piece of sheet metal is 45 cm wide, determine the angle θ that maximizes the volume of water that the trough can hold.
What is the cross-sectional area of this maximal trough?



Solution

In terms of the variables in the question, the area of the cross-section is equal to

$$A = 2\left(\frac{1}{2}bh\right) + xh = bh + xh.$$

Since we are asked to find θ that maximizes the area, we need an expression for the cross-sectional area in terms of only the variable θ.
We can use trigonometry to do so:

$$\sin(\theta) = \frac{h}{x} \implies h = x\sin(\theta)$$
$$\cos(\theta) = \frac{b}{x} \implies b = x\cos(\theta)$$

Substituting these expressions into the equation for the area, we get

$$\begin{aligned} A &= bh + xh \\ &= \left(x\cos(\theta)\right)\left(x\sin(\theta)\right) + x\left(x\sin(\theta)\right) \\ &= x^2\cos(\theta)\sin(\theta) + x^2\sin(\theta) \end{aligned}$$

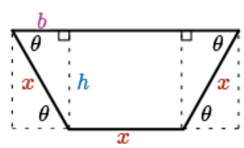
Since we are given that the sheet metal has width 45 cm, we have $x = \frac{1}{3}(45) = 15$ cm and so the area, as a function of θ only, is equal to

$$\begin{aligned} A(\theta) &= 15^2\cos(\theta)\sin(\theta) + 15^2\sin(\theta) \\ &= 225(\cos(\theta)\sin(\theta) + \sin(\theta)) \end{aligned}$$

Examples

Example 4

A company makes troughs for holding water for farm animals.
The frame of the trough is made by taking a piece of sheet metal and folding $\frac{1}{3}$ of the sheet up on each side, so that each side makes an angle θ with the ground.
Assuming that the original piece of sheet metal is 45 cm wide, determine the angle θ that maximizes the volume of water that the trough can hold.
What is the cross-sectional area of this maximal trough?



Solution

$$A(\theta) = 225(\cos(\theta)\sin(\theta) + \sin(\theta))$$

We will determine the absolute maximum value of the function $A(\theta)$, over the closed interval $\left[0, \frac{\pi}{2}\right]$, using the extreme value theorem. First, let's locate the critical points of the area function:

$$\begin{aligned} A'(\theta) &= 225\left[\cos(\theta)\frac{d}{d\theta}\left(\sin(\theta)\right) + \sin(\theta)\frac{d}{d\theta}\left(\cos(\theta)\right) + \cos(\theta)\right] \\ &= 225\left[\cos(\theta)\cos(\theta) + \sin(\theta)(-\sin(\theta)) + \cos(\theta)\right] \\ &= 225\left[\cos^2(\theta) - \sin^2(\theta) + \cos(\theta)\right] \end{aligned}$$

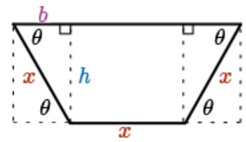
Therefore, $A'(\theta) = 0$ if and only if $\cos^2(\theta) - \sin^2(\theta) + \cos(\theta) = 0$.
To find the critical point(s), we need to solve this equation.

Examples

Example 4

A company makes troughs for holding water for farm animals.
The frame of the trough is made by taking a piece of sheet metal and folding $\frac{1}{3}$ of the sheet up on each side, so that each side makes an angle θ with the ground.

Assuming that the original piece of sheet metal is 45 cm wide, determine the angle θ that maximizes the volume of water that the trough can hold.
What is the cross-sectional area of this maximal trough?



Solution

$$\cos^2(\theta) - \sin^2(\theta) + \cos(\theta) = 0$$

First, we rewrite the expression $\cos^2(\theta) - \sin^2(\theta) + \cos(\theta)$ in terms of only $\cos(\theta)$ as follows:

$$\begin{aligned}\cos^2(\theta) - \sin^2(\theta) + \cos(\theta) &= \cos^2(\theta) - (1 - \cos^2(\theta)) + \cos(\theta)\\ &= 2\cos^2(\theta) + \cos(\theta) - 1\end{aligned}$$

This is a quadratic polynomial in $\cos(\theta)$, that is, if we let $u = \cos(\theta)$ then we have

$$2\cos^2(\theta) + \cos(\theta) - 1 = 2u^2 + u - 1$$

We can factor the polynomial $y = 2u^2 + u - 1 = (2u - 1)(u + 1)$, so we can factor the trigonometric expression as

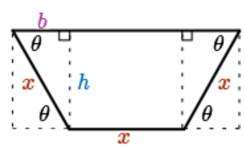
$$2\cos^2(\theta) + \cos(\theta) - 1 = (2\underbrace{\cos(\theta)}_{u} - 1)(\underbrace{\cos(\theta)}_{u} + 1)$$

Examples

Example 4

A company makes troughs for holding water for farm animals.
The frame of the trough is made by taking a piece of sheet metal and folding $\frac{1}{3}$ of the sheet up on each side, so that each side makes an angle θ with the ground.

Assuming that the original piece of sheet metal is 45 cm wide, determine the angle θ that maximizes the volume of water that the trough can hold.
What is the cross-sectional area of this maximal trough?



Solution

$$(2\cos(\theta) - 1)(\cos(\theta) + 1) = 0$$

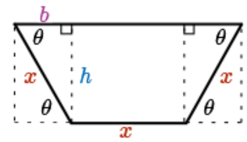
Therefore, $A'(\theta) = 0$ if and only if $\cos(\theta) = \frac{1}{2}$ or $\cos(\theta) = -1$.
There is no angle θ between 0 and $\frac{\pi}{2}$ that satisfies $\cos(\theta) = -1$, but there is one such angle that satisfies $\cos(\theta) = \frac{1}{2}$, namely $\theta = \frac{\pi}{3}$. So, there is exactly one critical point in the interval $\left[0, \frac{\pi}{2}\right]$: $\theta = \frac{\pi}{3}$.
Next, we evaluate the function at the critical point and the endpoints to locate the maximum value:

$$\begin{aligned}A(0) &= 225\Big(\cos(0)\sin(0) + \sin(0)\Big) = 225(1 \cdot 0 + 0) = 0\\ A\left(\frac{\pi}{3}\right) &= 225\left(\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right)\right) = 225\left(\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\right) \approx 292\\ A\left(\frac{\pi}{2}\right) &= 225\left(\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)\right) = 225(0 \cdot 1 + 1) = 225\end{aligned}$$

Examples

Example 4

A company makes troughs for holding water for farm animals.
The frame of the trough is made by taking a piece of sheet metal and folding $\frac{1}{3}$ of the sheet up on each side, so that each side makes an angle θ with the ground.
Assuming that the original piece of sheet metal is 45 cm wide, determine the angle θ that maximizes the volume of water that the trough can hold.
What is the cross-sectional area of this maximal trough?



Solution

Note:
We have $A(0) = 0$ and this is the absolute minimum value over the interval $\left[0, \frac{\pi}{2}\right]$.
An angle of $\theta = 0$ corresponds to the sides of the trough lying flat on the ground and so the cross-sectional area should be 0 in this case.
Comparing the function values, we have that $A(\theta)$ achieves its maximum value at $x = \frac{\pi}{3}$.

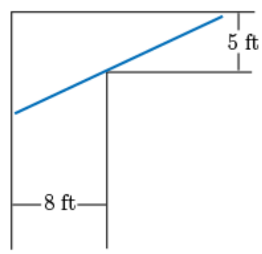
$$A(0) = 0$$
$$A\left(\frac{\pi}{3}\right) \approx 292$$
$$A\left(\frac{\pi}{2}\right) = 225$$

Therefore, the maximum cross-sectional area, and hence the maximum volume, occurs when $\theta = \frac{\pi}{3}$, and the maximum possible cross-sectional area is approximately 292 cm^2.

Examples

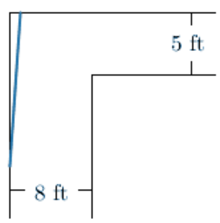
Challenge Problem

A pipe is to be carried down a hallway that is 8 ft wide. At the end of the hallway, there is a right-angled turn into a narrower hallway that is only 5 ft wide.
If the pipe needs to remain horizontal while carried, then what is the length of the longest pipe that can be carried around the corner?



Solution

Here are some important observations about the physical situation:
1. An optimal way to pass the pipe through the hallway is to slide it along the left wall of the first hallway, and turn it around the corner until it is parallel to the walls of the second hallway.

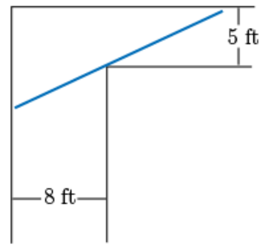


So, we will think about keeping one endpoint of the pipe in contact with the left wall of the first hallway for the entire turn.
We will call the angle that the pipe makes with this wall θ.

Examples

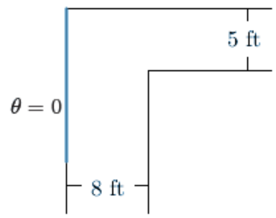
Challenge Problem

A pipe is to be carried down a hallway that is 8 ft wide. At the end of the hallway, there is a right-angled turn into a narrower hallway that is only 5 ft wide.
If the pipe needs to remain horizontal while carried, then what is the length of the longest pipe that can be carried around the corner?



Solution

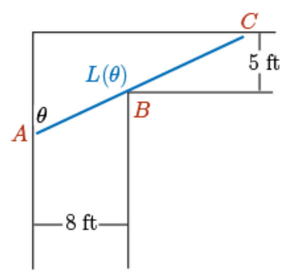
2. If the pipe manages to make the turn, then the angle θ (defined in part 1)) must pass from $\theta = 0$ to $\theta = \frac{\pi}{2}$, taking on every angle in between.



Examples

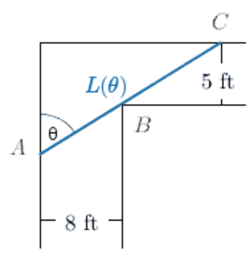
Challenge Problem

A pipe is to be carried down a hallway that is 8 ft wide. At the end of the hallway, there is a right-angled turn into a narrower hallway that is only 5 ft wide.
If the pipe needs to remain horizontal while carried, then what is the length of the longest pipe that can be carried around the corner?



Solution

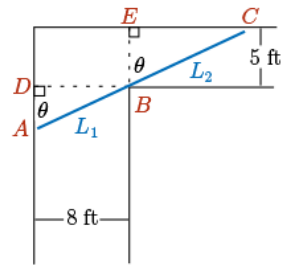
3. For each angle θ in the interval $\left(0, \frac{\pi}{2}\right)$, there is a maximum length, $L(\theta)$, of a pipe that can fit in the hallway while making an angle of θ with the left wall.
This maximum length is the length of the pipe that has the following configuration: tangent to the left wall at A, the inner corner at B, and the upper wall at C.



Examples

Challenge Problem

A pipe is to be carried down a hallway that is 8 ft wide. At the end of the hallway, there is a right-angled turn into a narrower hallway that is only 5 ft wide.
If the pipe needs to remain horizontal while carried, then what is the length of the longest pipe that can be carried around the corner?



Solution

4. For a pipe of length $\mathcal{L}$ to make the turn, we must have $\mathcal{L} \le L(\theta)$ for all $0 < \theta < \frac{\pi}{2}$.

If the function $L(\theta)$ has an absolute minimum value, say L_0, over the interval $\left(0, \frac{\pi}{2}\right)$, then any pipe that can make the turn must have length $\mathcal{L} \le L_0$.

It follows that the *longest* pipe that can make the turn is a pipe whose length is *equal* to this minimum length L_0.

Now, we will attempt to determine the value of L_0 – the absolute minimum of the function $L(\theta)$ over $\left(0, \frac{\pi}{2}\right)$.

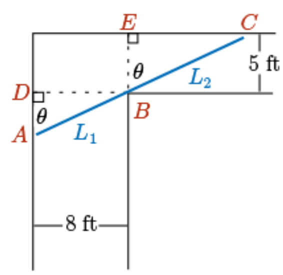
First, we can obtain an expression for $L(\theta) = L_1 + L_2$ using the above diagram.

$$\sin(\theta) = \frac{DB}{AB} = \frac{8}{L_1} \quad \Longrightarrow \quad L_1 = 8\csc(\theta)$$
$$\cos(\theta) = \frac{EB}{BC} = \frac{5}{L_2} \quad \Longrightarrow \quad L_2 = 5\sec(\theta)$$
$$L(\theta) = L_1 + L_2 = 8\csc(\theta) + 5\sec(\theta)$$

Examples

Challenge Problem

A pipe is to be carried down a hallway that is 8 ft wide. At the end of the hallway, there is a right-angled turn into a narrower hallway that is only 5 ft wide.
If the pipe needs to remain horizontal while carried, then what is the length of the longest pipe that can be carried around the corner?



Solution

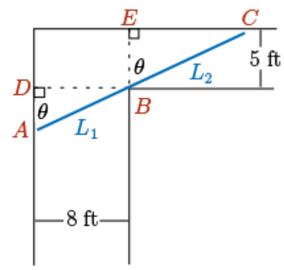
$$L(\theta) = 8\csc(\theta) + 5\sec(\theta)$$

We need to find the minimum value of $L(\theta)$.
Differentiating gives

$$L'(\theta) = 8\Big(-\csc(\theta)\cot(\theta)\Big) + 5\sec(\theta)\tan(\theta) \qquad \text{as } \frac{d}{dx}\Big(\csc(\theta)\Big) = -\csc(\theta)\cot(\theta)$$
$$= 5\sec(\theta)\tan(\theta) - 8\csc(\theta)\cot(\theta) \qquad \text{and } \frac{d}{dx}\Big(\sec(\theta)\Big) = \sec(\theta)\tan(\theta)$$

Examples

Challenge Problem



A pipe is to be carried down a hallway that is 8 ft wide. At the end of the hallway, there is a right-angled turn into a narrower hallway that is only 5 ft wide.
If the pipe needs to remain horizontal while carried, then what is the length of the longest pipe that can be carried around the corner?

Solution

Setting $L'(\theta) = 0$, gives

$$5\sec(\theta)\tan(\theta) - 8\csc(\theta)\cot(\theta) = 0$$

$$\therefore 5\sec(\theta)\tan(\theta) = 8\csc(\theta)\cot(\theta)$$

$$\frac{\sec(\theta)\tan(\theta)}{\csc(\theta)\cot(\theta)} = \frac{8}{5}$$

$$\frac{\sin(\theta)\tan(\theta)}{\cos(\theta)\cot(\theta)} = \frac{8}{5}$$

$$\tan^3(\theta) = \frac{8}{5}$$

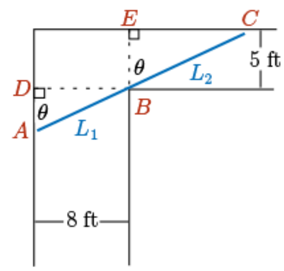
$$\tan(\theta) = \sqrt[3]{\frac{8}{5}}$$

Since we are working over the interval $0 < \theta < \frac{\pi}{2}$, we have $\theta = \tan^{-1}\left(\sqrt[3]{\frac{8}{5}}\right) \approx 0.863413605$.

Let's denote this angle by θ_0.

Examples

Challenge Problem



A pipe is to be carried down a hallway that is 8 ft wide. At the end of the hallway, there is a right-angled turn into a narrower hallway that is only 5 ft wide.
If the pipe needs to remain horizontal while carried, then what is the length of the longest pipe that can be carried around the corner?

Solution

We can check that $L(\theta_0)$ is a local *minimum* of the function $L(\theta)$, and that it is the absolute minimum over the interval $0 < \theta < \frac{\pi}{2}$. (You should verify this by examining the sign of $L'(\theta)$ over this open interval).

Therefore, $L_0 = L(\theta_0)$.

We find $L(\theta_0) \approx L(0.863413605) \approx 18.21953370$ and so the longest pipe that can make the turn measures approximately 18.2 feet.