The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

# Differentiating Exponential Functions $f(x)=a^x$

## Introduction

Many exponential function models do not have Euler's number, $e$, as the base.

For example, in biological models, $2$ is often used as the base for exponential functions because it provides an easy way to represent a population's "doubling time."

Another example is the decay of radioactive substances, which is usually expressed by the half-life, leading mathematicians to use a base of $\frac{1}{2}$.

## The Derivative of Exponential Functions: $y = a^x$

Let $y = a^x$ where $a > 0$ is a constant and $a \in \mathbb{R}$.

We begin by taking the $\ln$ of both sides which yields

$$\begin{aligned} \ln(y) &= \ln(a^x) \\ &= x\ln(a) \end{aligned}$$

Since $a$ is a positive constant, this equation could be written as $\ln(y) = (\ln(a))x$.

Implicitly differentiating both sides with respect to $x$ gives

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \ln(a)(1)$$

Solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = y(\ln(a))$$

Since $y = a^x$,

$$\frac{dy}{dx} = a^x \ln(a)$$

## The Derivative of Exponential Functions: $y = a^x$

**The Derivative of Exponential Functions: $f(x) = a^x$**

If $f(x) = a^x$, where $a$ is a positive constant, then $f'(x) = a^x \ln(a)$ for all real $x$.

**Note:**

The general form for the derivative of any exponential function will work if $a = e$.

If $f(x) = e^x$, then the general form will give $f'(x) = e^x \ln(e)$ and since $\ln(e) = 1$, we have $f'(x) = e^x(1) = e^x$.

## Examples

### Example 1

Differentiate $y = 3^x$.

**Solution**

$$\frac{dy}{dx} = 3^x \ln(3)$$

## Examples

### Example 2

Differentiate $f(x) = (4^x)(5^x)$.

**Solution**

**Method 1: Product Rule**

Since we have a product of two functions $g(x) = 4^x$ and $h(x) = 5^x$, let's use the product rule to differentiate.

$$\begin{aligned} f'(x) &= g'(x)[h(x)] + g(x)[h'(x)] \\ &= (4^x \ln(4))5^x + 4^x(5^x \ln(5)) \\ &= 4^x 5^x \Big(\ln(4) + \ln(5)\Big) && \text{removing common factors} \\ &= (4 \cdot 5)^x \Big[\ln(4 \cdot 5)\Big] && \text{applying laws of logarithms} \\ &= 20^x \ln(20) \end{aligned}$$


## Examples

### Example 2

Differentiate $f(x) = (4^x)(5^x)$.

**Solution**


**Method 2: Applying Exponent Laws to Simplify Before Differentiating**

$$\begin{aligned} y &= 4^x 5^x \\ &= [4(5)]^x \\ &= 20^x \\ \therefore \frac{dy}{dx} &= 20^x(\ln(20)) \end{aligned}$$

## Examples

### Example 3

Differentiate $y = 3^{\ln(x)}$.

**Solution**

**Method 1: Chain Rule**

The function $y = 3^{\ln(x)}$ is a composite function.
The inner function is $u = \ln(x)$ while the outer function is $y = 3^u$.
To differentiate this function, we must apply the chain rule.

$$\begin{aligned}\frac{dy}{dx} &= \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)\\ &= 3^u(\ln(3))\left(\frac{1}{x}\right)\\ &= \frac{3^{\ln(x)}(\ln(3))}{x}\end{aligned}$$

## Examples

### Example 3

Differentiate $y = 3^{\ln(x)}$.

**Solution**

**Method 2: Converting the Exponential Function to Have Base $e$**

Recall that $e^{\ln(a)} = a, a > 0$.
So $3 = e^{\ln(3)}$ and the function $y = 3^{\ln(x)}$ can be written as

$$y = \left(e^{\ln(3)}\right)^{\ln(x)} = e^{\ln(3)\ln(x)}$$

Differentiating the expression by applying chain rule, we have

$$\begin{aligned}\frac{dy}{dx} &= e^{\ln(3)\ln(x)}\left[\ln(3)\left(\frac{1}{x}\right)\right]\\ &= \frac{3^{\ln(x)}\ln(3)}{x}\end{aligned}$$

## Examples

### Challenge Question

A radioactive material exponentially decays by $20\%$ each year. The original mass of the substance was $100$ g.

**a.** Determine a mathematical model to represent the decay of this radioactive substance.

**b.** What is the half-life of this substance?

**c.** How fast is the substance decaying at the instant when exactly half of the substance remains (that is, after one half-life)?

**Solution**

**a.** Since the radioactive substance decays by $20\%$ per year, it retains $80\%$ of its radioactive mass.
So, after $1$ year, there would be $0.8 \times 100$ g.
After $2$ years, we would have $0.8(0.8 \times 100) = 0.8^2 \times 100$ g, and so on.
Thus, the mass, $M$, of the radioactive substance $t$ years after it begins to decay is modelled by the exponential equation

$$M(t) = 100(0.8)^t$$


## Examples

### Challenge Question

A radioactive material exponentially decays by $20\%$ each year. The original mass of the substance was $100$ g.

**a.** Determine a mathematical model to represent the decay of this radioactive substance.

**b.** What is the half-life of this substance?

**c.** How fast is the substance decaying at the instant when exactly half of the substance remains (that is, after one half-life)?

**Solution**


**b.** The original mass of the radioactive substance was $100$ g.
The half-life of the substance is the time required for exactly half of the radioactive substance to decay.
In this case, this is the time it takes for the substance to be reduced from $100$ g to $\frac{1}{2}(100) = 50$ g.
To determine when this will happen, we substitute $M = 50$ into the mathematical model and solve for $t$.

$$\begin{aligned} M(t) &= 100(0.8)^t \\ 50 &= 100(0.8)^t \\ \frac{1}{2} &= (0.8)^t \end{aligned}$$

Taking the $\ln$ of both sides,

$$\begin{aligned} \ln\left(\frac{1}{2}\right) &= \ln\left(0.8^t\right) \\ \ln\left(2^{-1}\right) &= \ln\left(0.8^t\right) \\ -\ln(2) &= t(\ln(0.8)) \\ t &= \frac{-\ln(2)}{\ln(0.8)} \approx 3.106 \text{ years} \end{aligned}$$

Therefore, the half-life of this radioactive substance is approximately $3.106$ years.

## Examples

### Challenge Question

A radioactive material exponentially decays by $20\%$ each year. The original mass of the substance was $100$ g.

**a.** Determine a mathematical model to represent the decay of this radioactive substance.

**b.** What is the half-life of this substance?

**c.** How fast is the substance decaying at the instant when exactly half of the substance remains (that is, after one half-life)?

#### Solution

**c.** We need to determine the rate of change of the decay of the radioactive substance at $t = 3.106$ years.

Let's begin by differentiating the function $M(t) = 100(0.8)^t$ which will give a function for the rate of change of decay, namely

$$M'(t) = 100(0.8)^t \ln(0.8)$$

Now, substitute $t = 3.106$ and solve for $M'(t)$.

$$M'(3.106) = 100(0.8)^{3.106} \ln(0.8) \approx -11.2 \text{ g/year}$$

Therefore, after approximately one half-life, the radioactive substance is decaying at a rate of about $11.2$ g per year.