Euler’s Number, e, and the Natural Logarithm

Introduction

Leonhard Euler (1707-1783) was a remarkable Swiss mathematician and physicist.

He made massive contributions to mathematics, especially calculus, as well as physics, optics, magnetism, astronomy, and shipbuilding.

Euler popularized the use of the symbol π and developed new approximations for it.

He was the first to use the symbol i to represent imaginary numbers.

Euler also developed the irrational number e, which is known as Euler's number and is defined as a limit:

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

Euler's Number e:

Let's examine some integer values of x to see the limiting value of this expression.

x	1	2	3	...
$y=\left(1+\frac{1}{x}\right)^x$	$\left(1+\frac{1}{1}\right)^1=\left(\frac{2}{1}\right)^1$ $=2$	$\left(1+\frac{1}{2}\right)^2=\left(\frac{3}{2}\right)^2$ $=2.25$	$\left(1+\frac{1}{3}\right)^3=\left(\frac{4}{3}\right)^3$ ≈ 2.3704	... ...

...	100	1000	10000
... ...	$\left(1+\frac{1}{100}\right)^{100}=\left(\frac{101}{100}\right)^{100}$ ≈ 2.7048	$\left(1+\frac{1}{1000}\right)^{1000}=\left(\frac{1001}{1000}\right)^{1000}$ ≈ 2.7169	$\left(1+\frac{1}{10\,000}\right)^{10\,000}=\left(\frac{10\,001}{10\,000}\right)^{10\,000}$ ≈ 2.7181

We see that as x becomes larger, the value of the expression changes by a smaller and smaller amount.

In fact, the change can be shown to approach 0.

In other words, the expression is approaching a limiting value.

The limiting value of this expression is the irrational number $e = 2.718281828459\ldots$, a non-terminating decimal.

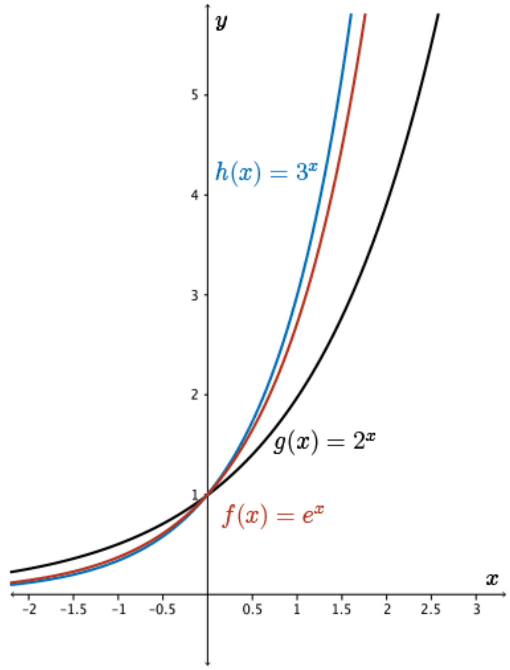
By making the substitution $t=\frac{1}{x}$, we get the following alternate definition of Euler's number:

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{t\to 0}(1+t)^{\frac{1}{t}}$$

Exponential Functions with Base e

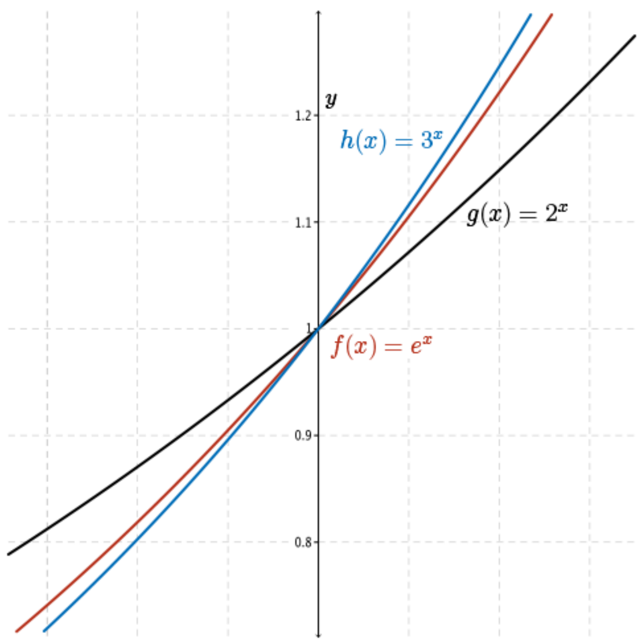
Recall the graph of the basic exponential function $y = 2^x$.

Since Euler's number is between 2 and 3, let's compare the graph of $f(x) = e^x$ with the graphs of $g(x) = 2^x$ and $h(x) = 3^x$.



Exponential Functions with Base e

Special Note: Included are the graphs of 2^x, e^x, and 3^x zoomed in at $x = 0$. Let's carefully examine the slope of the tangent to each of these graphs at $x = 0$. It is worth noting that the slope of the tangent line to 2^x is less than 1, while the slope of the tangent line to 3^x is greater than 1.



Exponential Functions with Base e

Special Note: Included are the graphs of 2^x, e^x, and 3^x zoomed in at $x = 0$.
Let's carefully examine the slope of the tangent to each of these graphs at $x = 0$.
It is worth noting that the slope of the tangent line to 2^x is less than 1, while the slope of the tangent line to 3^x is greater than 1.
Euler's number, e, is the base needed to make an exponential function have exactly slope 1 at $x = 0$.
We will use this definition of e when we discuss the derivative of the function $f(x) = e^x$.

The Inverse of the Exponential Function, e^x

In previous mathematics courses, you learned that the inverse of an exponential function is the logarithmic function with the same base.

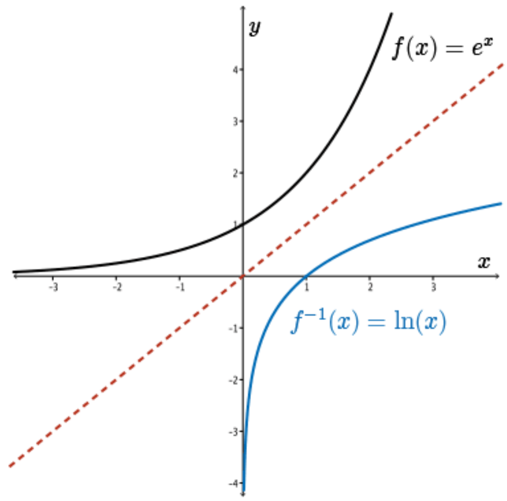
For example, the inverse of the exponential function $f(x) = 2^x$ is $f^{-1}(x) = \log_2(x)$.
The inverse of the exponential function $g(x) = e^x$ is $g^{-1}(x) = \log_e(x) = \ln(x)$.

Rather than using $\log_e(x)$, mathematicians use $\ln(x)$ to shorten this expression.
$\ln(x)$ stands for the natural logarithm of x and is pronounced “lawn x.”

The Graphs of the Exponential Function $y = e^x$ and the Logarithmic Function $y = \ln(x)$

An exponential equation can be converted into a logarithmic equation.

For example, if $e^x = c$, where c is some positive constant, then $x = \ln(c)$.
Converting exponential equations into logarithmic equations can be very useful when solving equations.
You will find $\ln$ as a button on your scientific calculator.



The Derivative of the Exponential Function, e^x

Let's investigate the slope of the tangent to many points along the curve of $f(x) = e^x$ by using the following Maple investigation.
In this worksheet, you will see the graph of $f(x) = e^x$ and a tangent drawn at one point on the left side of the graph.
Maple has calculated the slope of this tangent and then plotted the x-coordinate of the point of tangency with the value of the slope of the tangent at that point.
This will give us the numerical values of the derivative of $f(x) = e^x$.

If we were to connect these values, what function would you see?
Try it now.

The Derivative of the Exponential Function, $f(x) = e^x$

From the worksheet, we see that the slope of the tangent to the curve is equal to the y-value at each point of tangency. What does this tell us? This tells us that for $f(x) = e^x$, $f'(x) = e^x$ as well.
Using the definition of the derivative to differentiate the function $f(x) = e^x$ leads to another interesting limit.

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x e^h - e^x}{h}\\ &= \lim_{h\to 0}\left[e^x\left(\frac{e^h-1}{h}\right)\right]\\ &= \left(\lim_{h\to 0}e^x\right)\left(\lim_{h\to 0}\frac{e^h-1}{h}\right) \qquad \text{by using limit properties}\end{aligned}$$

Recall that Euler's number, e, is the base needed to make an exponential function have slope exactly 1 at $x = 0$.
Therefore, the value of the limit $\lim\limits_{h\to 0}\frac{e^h-1}{h}$ must be 1 by this definition of e, since this limit is exactly the definition of the derivative of e^x at 0.
You may study this limit in future mathematics courses.

The Derivative of the Exponential Function, $f(x) = e^x$

$$\frac{dy}{dx} = \left(\lim_{h\to 0}e^x\right)\underbrace{\left(\lim_{h\to 0}\frac{e^h-1}{h}\right)}_{=1}$$

Also, since e^x does not depend on h, we know that

$$\left(\lim_{h\to 0}e^x\right) = e^x$$

Thus,

$$\begin{aligned}\frac{dy}{dx} &= (e^x)(1)\\ &= e^x\end{aligned}$$

Therefore, e^x has the remarkable property that

$$\frac{d}{dx}\left(e^x\right) = e^x$$

The Derivative of the Exponential Function $f(x) = e^x$

If $f(x) = e^x$, then $f'(x) = e^x$ for all real x.

Examples

Example 1

Differentiate $y = e^{4x}$.

Solution

The function $y = e^{4x}$ is a composite function.
The inner function is $g(x) = 4x$, while the outer function is $h(x) = e^x$.
To differentiate this function, we must use the chain rule.
If we let $u = 4x$, then $y = e^u$.

$$\begin{aligned}\frac{dy}{dx} &= \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)\\ &= (e^u)(4)\\ &= 4e^{4x}\end{aligned}$$

Examples

Example 2

Differentiate $f(x) = x^5 e^{-3x^2}$.

Solution

The function $f(x) = x^5 e^{-3x^2}$ is a product of the 2 functions $g(x) = x^5$ and $h(x) = e^{-3x^2}$.
Therefore, to differentiate this function, we must use the product rule.
Also, the function $h(x) = e^{-3x^2}$ is a composite function, so to differentiate this function, we must use the chain rule.

Let's begin by differentiating $h(x) = e^{-3x^2}$.
If we let $u = -3x^2$, then $y = e^u$.

$$\begin{aligned}\frac{dy}{dx} &= \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)\\ &= (e^u)(-6x)\\ &= -6xe^{-3x^2}\end{aligned}$$

Now let's differentiate the entire function,
$f(x) = x^5 e^{-3x^2}$.

$$\begin{aligned}f'(x) &= g'(x)h(x) + g(x)h'(x)\\ &= (5x^4)(e^{-3x^2}) + (x^5)(-6xe^{-3x^2})\\ &= 5x^4e^{-3x^2} - 6x^6e^{-3x^2}\\ &= x^4e^{-3x^2}(5 - 6x^2) \quad \text{remove common factors}\end{aligned}$$

Special Note: It is important to factor the derivative as much as possible, as this will help when sketching the graph of the function or determining its properties.

Examples

Example 3

Differentiate $y = e^x(e^{4x})$.

Solution

Method 1

Differentiate with the product rule and chain rule:

$$\begin{aligned}\frac{dy}{dx} &= e^x(e^{4x}) + e^x(e^{4x})(4) \\ &= 5e^x(e^{4x}) \\ &= 5e^{5x}\end{aligned}$$

Method 2

Apply exponent rules to simplify before differentiating:

$$\begin{aligned}y &= e^x(e^{4x}) \\ &= e^{x+4x} \\ &= e^{5x}\end{aligned}$$

Now, let's differentiate this function using the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= (e^{5x})(5) \\ &= 5e^{5x}\end{aligned}$$

Examples

Challenge Question

Show that $e^{2x} + 4e^{-4x} > 2$ for all x.

Solution

To prove that $e^{2x} + 4e^{-4x} > 2$, we show that the minimum value of $e^{2x} + 4e^{-4x}$ is greater than 2.
Let $y = e^{2x} + 4e^{-4x}$.

First note that the domain of the function is $x \in \mathbb{R}$.
We also note that each of the exponentials e^{2x} and $4e^{-4x}$ are continuous and differentiable for all $x \in \mathbb{R}$ and so $y = e^{2x} + 4e^{-4x}$ is a continuous and differentiable function.

Therefore, if the function has an absolute minimum, then it must occur at local extreme where $\frac{dy}{dx} = 0$.

Differentiating $y = e^{2x} + 4e^{-4x}$,

$$\frac{dy}{dx} = 2e^{2x} - 16e^{-4x}$$

Examples

Challenge Question

Show that $e^{2x} + 4e^{-4x} > 2$ for all x.

Solution

$$\frac{dy}{dx} = 2e^{2x} - 16e^{-4x}$$

Since the minimum value will occur when $\frac{dy}{dx} = 0$, we solve for x.

$$\begin{aligned} 0 &= 2e^{2x} - 16e^{-4x} \\ 0 &= 2\left(e^{2x} - 8e^{-4x}\right) \\ 0 &= 2e^{-4x}\left[e^{6x} - 8\right] \end{aligned}$$

Therefore, either $e^{-4x} = 0$ or $e^{6x} - 8 = 0$.

But we know that $e^{-4x} > 0$ for all $x \in \mathbb{R}$.

Thus, we need

$$\begin{aligned} e^{6x} - 8 &= 0 \\ e^{6x} &= 8 \\ 6x &= \ln(8) \\ x &= \frac{1}{6}\ln(8) \approx 0.347 \end{aligned}$$

Examples

Challenge Question

Show that $e^{2x} + 4e^{-4x} > 2$ for all x.

Solution

Now, to determine the minimum value, let $x = \frac{1}{6}\ln(8)$ in the equation $y = e^{2x} + 4e^{-4x}$.

$$\begin{aligned} y &= e^{2\left(\frac{1}{6}\ln(8)\right)} + 4e^{-4\left(\frac{1}{6}\ln(8)\right)} \\ &= e^{\left(\frac{1}{3}\ln(8)\right)} + 4e^{\left(-\frac{2}{3}\ln(8)\right)} \\ &= \left(e^{\ln(8)}\right)^{\frac{1}{3}} + 4\left(e^{\ln(8)}\right)^{-\frac{2}{3}} \end{aligned}$$

Recall that $a^{\log_a(b)} = b$, so $e^{\ln(8)} = e^{\log_e(8)} = 8$. Thus,

$$\begin{aligned} y &= 8^{\frac{1}{3}} + 4(8)^{-\frac{2}{3}} \\ &= 2 + \frac{4}{\left(8^{\frac{1}{3}}\right)^2} \\ &= 3 \end{aligned}$$

Since the minimum value of $y = e^{2x} + 4e^{-4x}$ is 3, then $e^{2x} + 4e^{-4x} > 2$ for all x.