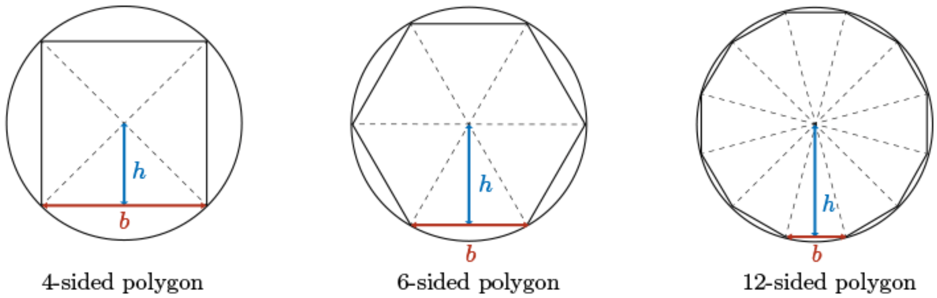
The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Introduction to Calculus: Limits and Rates Of Change

Developing a Formula for the Area of a Circle

The Greek Method of Exhaustion (300-200 BC)

Greek mathematicians inscribed regular polygons within a circle.



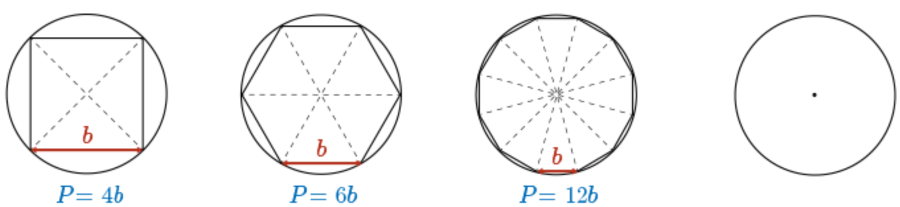
They divided each n-sided polygon into n congruent triangles.
Each triangle has base, b, and height, h.
The area of each polygon can be found by multiplying the area of the triangles by n.

Developing a Formula for the Area of a Circle

The Greek Method of Exhaustion (300-200 BC)

As the number of sides in the polygon increases, the perimeter, P, of each polygon approaches (becomes closer in value to) the circumference of the circle $(2\pi r)$.
The perimeter, P, of each polygon is calculated by multiplying the the number of sides in the polygon, n, by the length of the base of each triangle, b.

$$P = nb \approx 2\pi r$$



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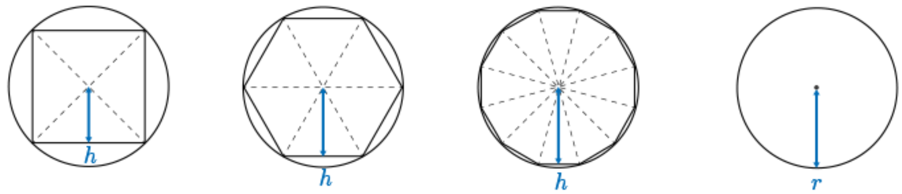
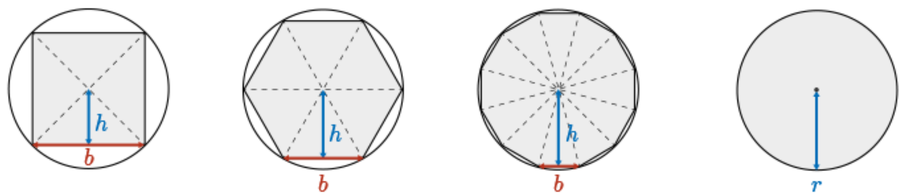
Also, as the number of sides in the polygon increases, the height, h, of each triangle approaches the length of the radius of the circle.



As the number of sides in the polygon increases, the area of the polygon approaches the area of the circle.



$$h \approx r$$

Developing a Formula for the Area of a Circle

The Greek Method of Exhaustion (300-200 BC)

To calculate the area, A_n, of the n-sided polygon, calculate the area, $A_{triangle}$, of one triangular section and multiply this by the number of triangular sections in the polygon. Thus,

$$\begin{aligned} A_n &= nA_{triangle} \\ &= n\left(\frac{1}{2}bh\right) \\ &= (nb)\left(\frac{1}{2}h\right) \\ &\approx (2\pi r)\left(\frac{1}{2}r\right) \qquad \text{recall that } P = nb \approx 2\pi r \text{ and } h \approx r \\ A_n &\approx \pi r^2 \end{aligned}$$

The area, A_n, of the polygon approaches the area of the circle, $A_n \approx \pi r^2$, and this approximation becomes more accurate as n increases.

Developing a Formula for the Area of a Circle

The Greek Method of Exhaustion (300-200 BC)

This does not mean that these areas will ever be equal. Instead, the area of the polygon becomes closer in value to the area of the circle as the number of sides in the polygon increases.

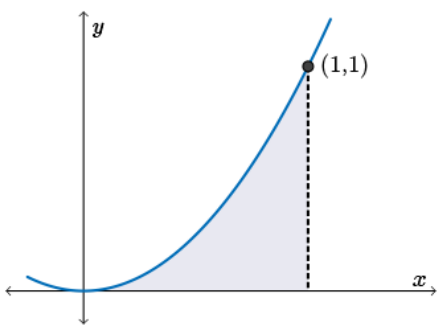
Mathematicians use the term **limit** to identify approached values and they use the following expression to represent this:

$$\lim_{n \to \infty} A_n = A_{Circle}$$

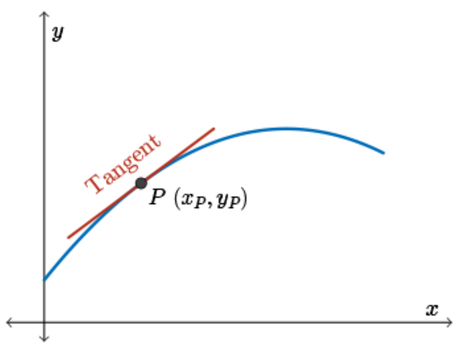
where n is the number of sides in the polygon.

Two Fundamental Problems of Calculus

The Area Problem

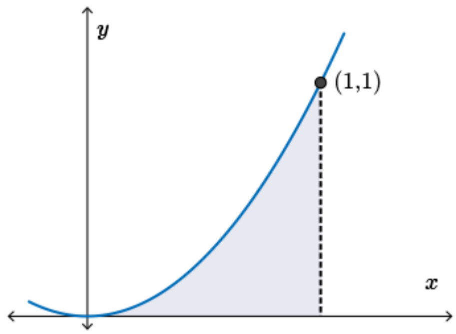


The Tangent Problem



Two Fundamental Problems of Calculus

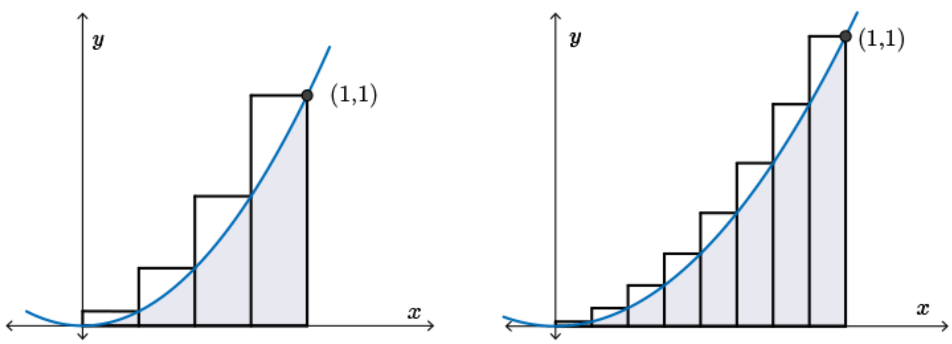
The Area Problem (Integral Calculus)



Mathematicians wanted to find the area under a curve. This is a very tough problem!

Two Fundamental Problems of Calculus

The Area Problem (Integral Calculus)



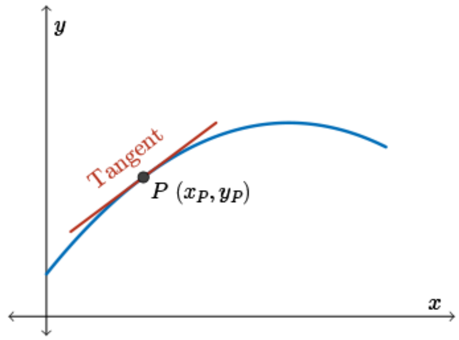
Mathematicians wanted to find the area under a curve. This is a very tough problem!
They used the sum of the areas of rectangles to help approximate the area under a curve.
As the width of the rectangle decreases (becomes very close to 0), the sum of the areas of the rectangles, A_R, approaches the area under the curve, A_C.
This is written as

$$\lim_{w \to 0} A_R = A_C$$

which means that as the width of the rectangles becomes almost 0, the sum of the area of the rectangles approaches the area under the curve.

Two Fundamental Problems of Calculus

The Tangent Problem (Differential Calculus)



Two Fundamental Problems of Calculus

The Tangent Problem (Differential Calculus)

A **tangent** is a line which tells us how "steep" the curve is at the point of tangency.

A tangent line may just "kiss" the curve, as at points P and Q in diagram 1.

A tangent line may cross through the curve, as at point P in diagram 2, where the curve changes from bending downwards to upwards or vice versa.

A curve may not always have a tangent line at each point, as at points P and Q in diagram 3, where the curve has a "sharp" point or "corner."

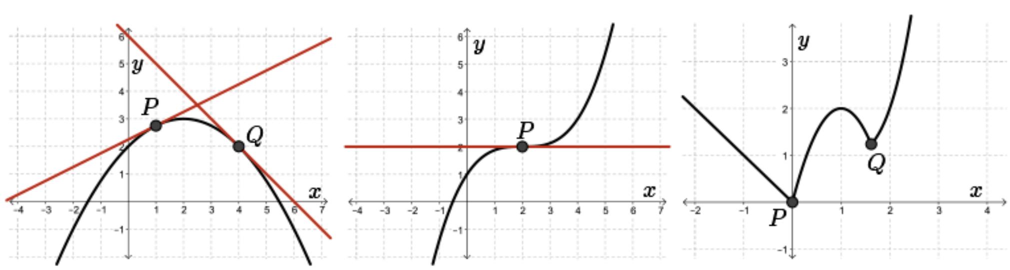
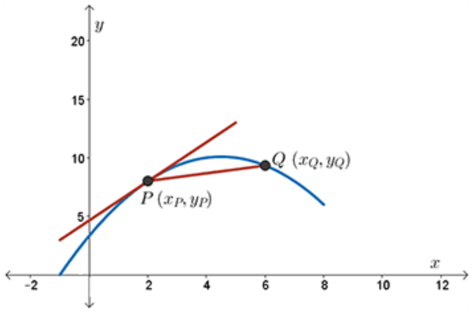


diagram 1 diagram 2 diagram 3

Two Fundamental Problems of Calculus

The Tangent Problem (Differential Calculus)

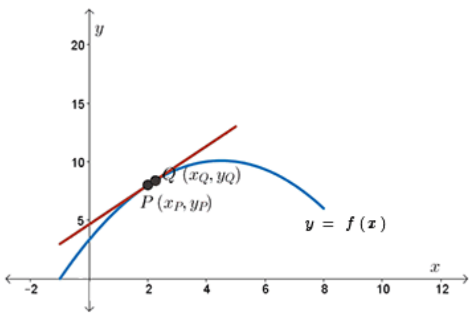
Before the invention of calculus, mathematicians wanted to find the equation of a tangent to a curve.
To find the equation of the tangent, we must calculate the slope of the tangent, which is a challenging problem.



However, the slope of the tangent can be approximated by examining the slopes of the secants nearby this point.
Choose another point, Q, on the curve that is nearby to P.
Draw a secant between points P and Q and calculate the slope of this secant.

Two Fundamental Problems of Calculus

The Tangent Problem (Differential Calculus)



As we move point Q towards point P and calculate the slope of each secant line, we notice that the slope of the secant approaches a certain value, which we take to be the slope of the tangent line.

$$\begin{aligned} m_{PQ} &= \frac{y_Q - y_P}{x_Q - x_P} \\ &= \frac{f(x_Q) - f(x_P)}{x_Q - x_P} \end{aligned}$$

$$\lim_{Q \to P} m_{PQ} = m_{tangent}$$

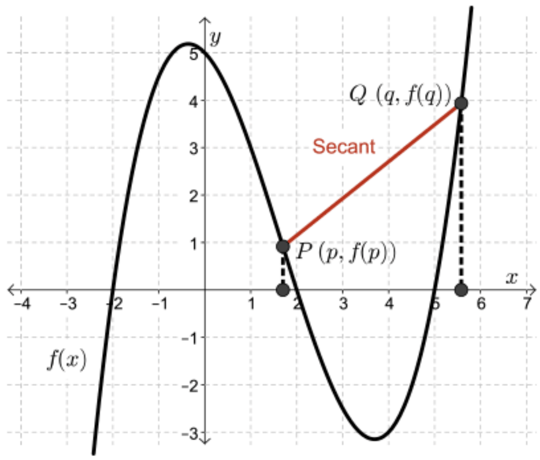
Rates of Change

Average Rate of Change

The average rate of change of a function, $f(x)$, over an interval, $p \leq x \leq q$, is defined as

$$Rate_{average} = \frac{f(q) - f(p)}{q - p}$$

which is the slope of the secant PQ.



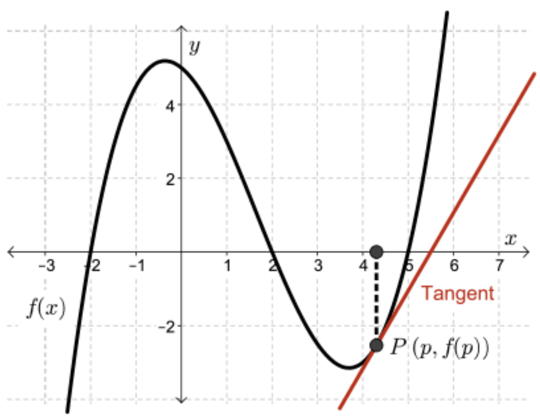
Rates of Change

Instantaneous Rate of Change

As we have seen, the slope of the tangent at point P is the limit of the slope of the secant between points $P\ (p, f(p))$ and $Q\ (q, f(q))$.

We define this limit as the instantaneous rate of change of $f(x)$ at $x = p$.

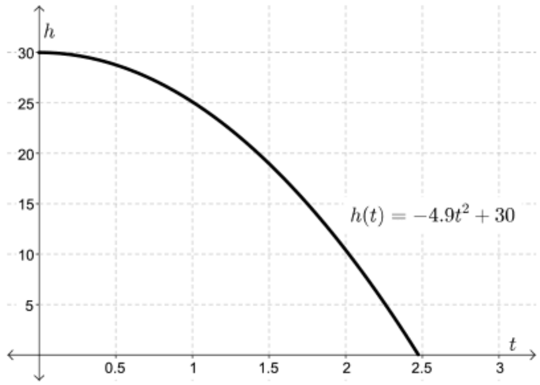
Geometrically,



Rates of Change

Application of Rates of Change

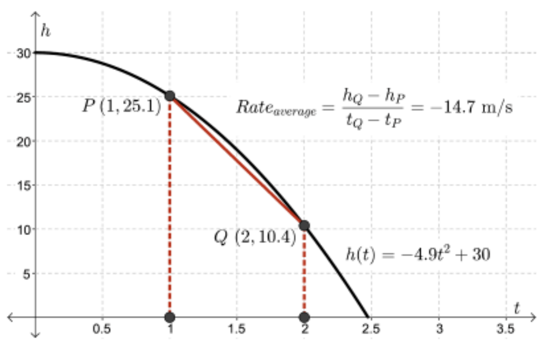
The following graph shows the height, $h(t)$, of a pebble above the surface of a river as a function of time, t, after it is dropped from a bridge.



Rates of Change

Application of Rates of Change

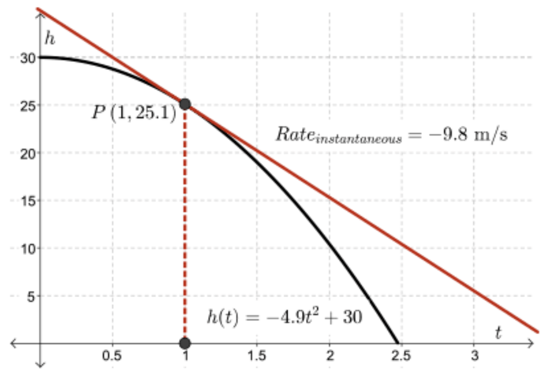
After the pebble is dropped, we can calculate the average speed of the falling pebble between 1 and 2 seconds by calculating the slope of the secant, PQ.



Rates of Change

Application of Rates of Change

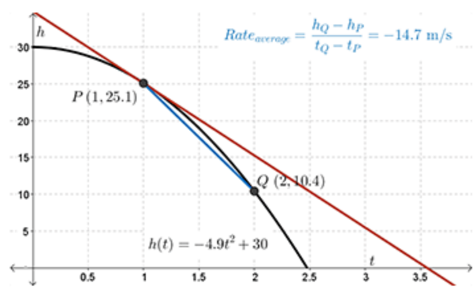
The instantaneous speed of the pebble exactly 1 second after it is dropped is found by calculating the slope of the tangent at point P.



Rates of Change

Application of Rates of Change

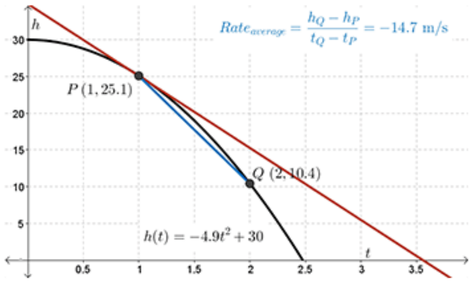
The slope of the secant between points P and Q can be used to approximate the slope of the tangent.



Rates of Change

Application of Rates of Change

Let's begin with point Q at $(2, 10.4)$. The slope of the secant, PQ, is -14.7.
Moving point Q towards point P at intervals of 0.1, the slope of the secant becomes closer in steepness to the slope of the tangent.
The following table of values tracks the slope of the secant as point Q moves closer to point P.



Δt $(t_Q - t_P)$	Δh $(h_Q - h_P)$	$m_{PQ} = \frac{\Delta h}{\Delta t}$
1	-14.7	-14.7

Rates of Change

Application of Rates of Change

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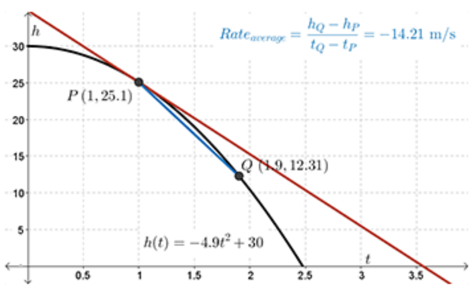

Δt $(t_Q - t_P)$	Δh $(h_Q - h_P)$	$m_{PQ} = \frac{\Delta h}{\Delta t}$
1	-14.7	-14.7
0.9	-12.789	-14.21

Rates of Change

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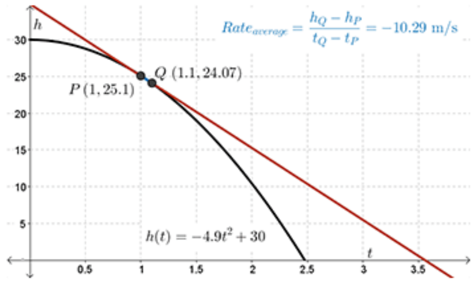
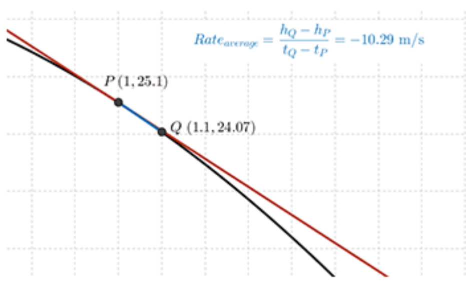


Δt $(t_Q - t_P)$	Δh $(h_Q - h_P)$	$m_{PQ} = \frac{\Delta h}{\Delta t}$
1	-14.7	-14.7
0.9	-12.789	-14.21
0.8	-10.976	-13.72
0.7	-9.261	-13.23
0.6	-7.644	-12.74
0.5	-6.125	-12.25
0.4	-4.704	-11.76
0.3	-3.381	-11.27
0.2	-2.156	-10.78
0.1	-1.029	-10.29

Rates of Change

Application of Rates of Change

To get a better approximation, let's zoom in on the graph and move point Q towards point P at intervals of 0.01 until point Q is just right of point P.



Δt $(t_Q - t_P)$	Δh $(h_Q - h_P)$	$m_{PQ} = \frac{\Delta h}{\Delta t}$
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Rates of Change

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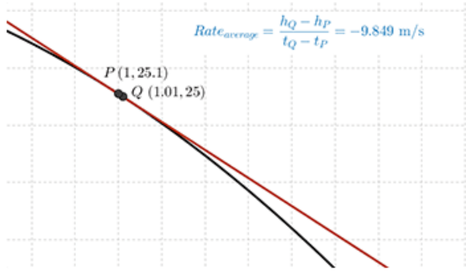


We now have values for the slope of the secant, PQ, as point Q approaches point P on the right side.

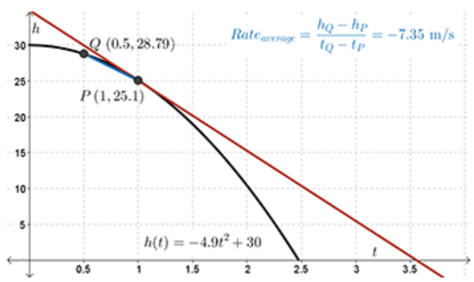
Δt $(t_Q - t_P)$	Δh $(h_Q - h_P)$	$m_{PQ} = \frac{\Delta h}{\Delta t}$
0.1	-1.029	-10.29
0.09	-0.92169	-10.241
0.08	-0.81536	-10.192
0.07	-0.71001	-10.143
0.06	-0.60564	-10.094
0.05	-0.50225	-10.045
0.04	-0.39984	-9.996
0.03	-0.29841	-9.947
0.02	-0.19796	-9.898
0.01	-0.09849	-9.849

Rates of Change

Application of Rates of Change

Let's now choose a point Q to the left of point P to observe how the slope of the secant approaches the slope of the tangent from the left side.

Let's begin at $Q\,(0.5, 28.79)$ and move point Q towards point P at intervals of 0.1.

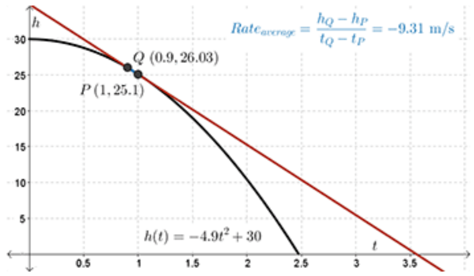


Δt $(t_Q - t_P)$	Δh $(h_Q - h_P)$	$m_{PQ} = \frac{\Delta h}{\Delta t}$
-0.5	3.675	-7.35

Rates of Change

Application of Rates of Change

Let's now choose a point Q to the left of point P to observe how the slope of the secant approaches the slope of the tangent from the left side.
Let's begin at $Q\ (0.5, 28.79)$ and move point Q towards point P at intervals of 0.1.

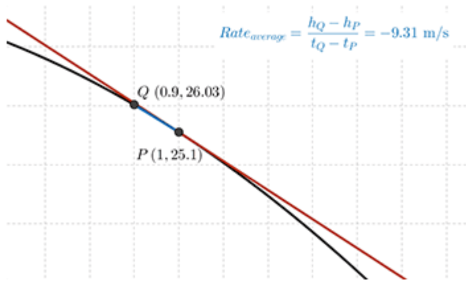


Δt $(t_Q - t_P)$	Δh $(h_Q - h_P)$	$m_{PQ} = \frac{\Delta h}{\Delta t}$
-0.5	3.675	-7.35
-0.4	3.136	-7.84
-0.3	2.499	-8.33
-0.2	1.764	-8.82
-0.1	0.931	-9.31

Rates of Change

Application of Rates of Change

Zooming in on the graph, let's move point Q towards point P at intervals of 0.01 until point Q is just left of point P.

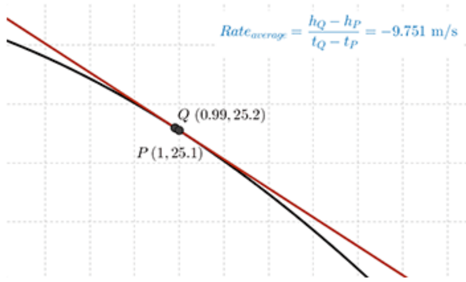


Δt $(t_Q - t_P)$	Δh $(h_Q - h_P)$	$m_{PQ} = \frac{\Delta h}{\Delta t}$
-0.1	0.931	-9.31

Rates of Change

Application of Rates of Change

Zooming in on the graph, let's move point Q towards point P at intervals of 0.01 until point Q is just left of point P.

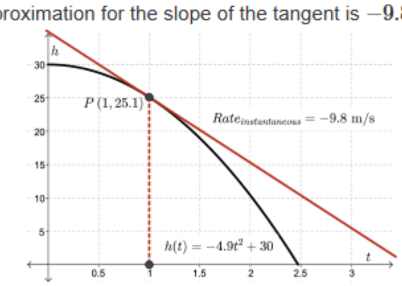


Δt $(t_Q - t_P)$	Δh $(h_Q - h_P)$	$m_{PQ} = \frac{\Delta h}{\Delta t}$
-0.1	0.931	-9.31
-0.09	0.84231	-9.359
-0.08	0.75264	-9.408
-0.07	0.66199	-9.457
-0.06	0.57036	-9.506
-0.05	0.47775	-9.555
-0.04	0.38416	-9.604
-0.03	0.28959	-9.653
-0.02	0.19404	-9.702
-0.01	0.09751	-9.751

Rates of Change

Application of Rates of Change

We now have the progression of slopes for secant PQ as Q approaches P from the left and right sides. By examining how the slopes of the secants change as we approach the middle of the table, our best approximation for the slope of the tangent is -9.8 m/s, as this value lies between -9.849 and -9.751.



Δt $(t_Q - t_P)$	Δh $(h_Q - h_P)$	$m_{PQ} = \frac{\Delta h}{\Delta t}$
0.07	-0.71001	-10.143
0.06	-0.60564	-10.094
0.05	-0.50225	-10.045
0.04	-0.39984	-9.996
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0.02	-0.19796	-9.898
0.01	-0.09849	-9.849
-0.01	0.09751	-9.751
-0.02	0.19404	-9.702
-0.03	0.28959	-9.653
-0.04	0.38416	-9.604
-0.05	0.47775	-9.555
-0.06	0.57036	-9.506
-0.07	0.66199	-9.457

Rates of Change

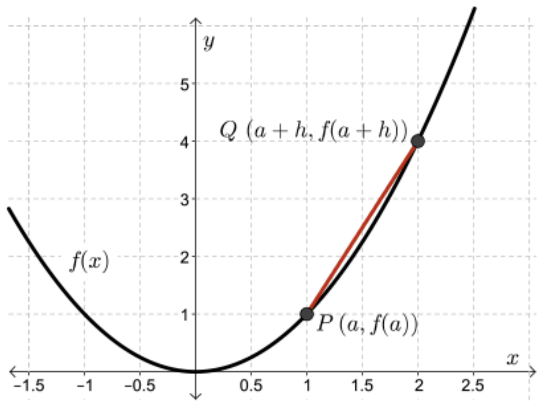
Calculating the Average Rate of Change for Any Function

The average rate of change between two points is calculated by finding the slope of the secant between these two points on the curve.

The point $P\ (a, f(a))$ is a point on the function $y = f(x)$.
A second point, Q, that is a horizontal displacement, h, away from P has coordinates $Q\ (a+h, f(a+h))$.
The slope of the secant is calculated by

$$m_{secant} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

and is called the **difference quotient**.



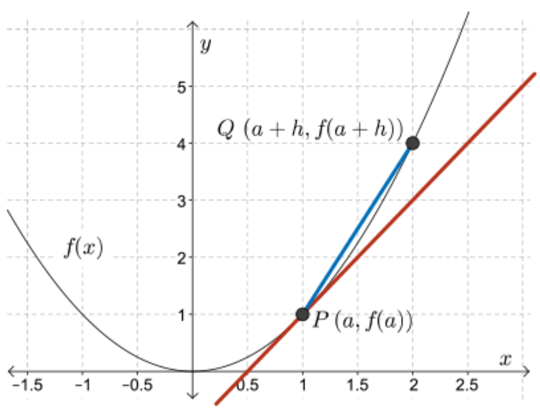
Rates of Change

Calculating the Instantaneous Rate of Change for Any Function

The instantaneous rate of change at a point is calculated from the slope of the tangent at that point.
We have seen that the slope of the tangent can be approximated by considering the slopes of the secants from the point of tangency to nearby points.
In general, to find the slope of the tangent to the function $y = f(x)$ at point $(a, f(a))$, choose a point that is a close horizontal displacement, h, away.
This point has coordinates $(a+h, f(a+h))$.
The slope of the tangent is found by considering the slopes of secants as point Q moves closer to point P.
In other words, the horizontal displacement between points Q and P approaches 0.
Thus, the slope is defined by

$$m_{tangent} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a}$$

the limit of the difference quotient.



Summary

Suppose $f(x)$ is defined on an open interval containing c, where Δy is defined to be $f(c+h)-f(c)$, and Δx is defined to be $(c+h)-c=h$. Then if

$$\lim_{h\to 0}\frac{\Delta y}{\Delta x}=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

exists, the line passing through $(c, f(c))$ with slope m, equal to this limit, is the tangent to the graph of $f(x)$ at the point $(c, f(c))$.

Examples

Example 1

Approximate the slope of the tangent to the curve $f(x)=\sqrt{x^2-25}$ at $x=8$.

Solution

Calculate the slopes of secants from point P where $x=8$ to points Q which are nearby, by adding and subtracting 0.1, 0.01, and 0.001 from $x=8$, to get values of x to the right and left of point P.
Use the following table to organize these values.

x_Q	$f(x_Q)=\sqrt{x_Q^2-25}$	Slope of Secant PQ
7.9		
7.99		
7.999		
8.001		
8.01		
8.1		

Examples

Example 1

Approximate the slope of the tangent to the curve $f(x) = \sqrt{x^2 - 25}$ at $x = 8$.

Solution

Use the following table to organize these values.

x_Q	$f(x_Q) = \sqrt{x_Q^2 - 25}$	Slope of Secant PQ
7.9		
7.99		
7.999		
8.001		
8.01		
8.1		

For this example, the slope of the secant is calculated by $m = \dfrac{f(x_Q) - f(8)}{x_Q - 8}$.

Examples

Example 1

Approximate the slope of the tangent to the curve $f(x) = \sqrt{x^2 - 25}$ at $x = 8$.

Solution

For this example, the slope of the secant is calculated by $m = \dfrac{f(x_Q) - f(8)}{x_Q - 8}$.

The completed table of values should be as follows:

x_Q	$f(x_Q) = \sqrt{x_Q^2 - 25}$	Slope of Secant PQ
7.9	6.11637147	1.28626525
7.99	6.2321826	1.281539517
7.999	6.24371692	1.281076564
8.001	6.24627897	1.280973918
8.01	6.25780313	1.28051305
8.1	6.37259759	1.275995881

By examining the table of values, we see that as point Q approaches point P from the left and right side, the slope of the secant appears to approach the value 1.281.

Examples

Example 2

Calculate the slope of the tangent to the curve $f(x) = x^2 - 6$ at $x = 2$.

Solution

If P is the point $(2, f(2))$ and Q is $(2+h, f(2+h))$, then

$$\begin{aligned} m_{secant} &= \frac{\Delta f(x)}{\Delta x} \\ &= \frac{f(x_Q) - f(x_P)}{x_Q - x_P} \\ &= \frac{((2+h)^2 - 6) - ((2)^2 - 6)}{(2+h) - 2} \\ &= \frac{((h^2 + 4h + 4) - 6) - (4 - 6)}{h} \\ &= \frac{h^2 + 4h}{h} \\ &= h + 4 \end{aligned}$$

Examples

Example 2

Calculate the slope of the tangent to the curve $f(x) = x^2 - 6$ at $x = 2$.

Solution

As point Q approaches P, the value h that was added to x becomes very small (i.e., approaches 0).
Thus, the slope of the tangent to any point on the function is found by

$$\begin{aligned} m_{tangent} &= \lim_{h \to 0} m_{secant} \\ &= \lim_{h \to 0} (h + 4) \\ &= (0) + 4 \\ &= 4 \end{aligned}$$