Methods of Evaluating Limits of Polynomial and Rational Functions

Introduction

There are many ways to evaluate a limit or show that it does not exist. Knowing which technique to use requires lots of practice; however, we will discuss some of the most useful methods. These methods include using the previously discussed limit properties as well as graphical techniques.

Evaluating Limits of Functions Which are Continuous for $x \in \mathbb{R}$

Consider the following limit:

$$ L = \lim_{x \to 2} 3x^2 $$

The graph of $f(x) = 3x^2$ is a parabola and since $f(x)$ is a polynomial function, it is continuous for all values of $x$. Therefore, the left-hand and right-hand limits exist and are equal to each other at any value of $x$ in the domain of the function, i.e., for all $x \in \mathbb{R}$.

The value of the given limit will then be

$$ L = \lim_{x \to 2} 3x^2 = f(2) = 3(2)^2 = 12 $$

In general, if a function $f(x)$ is continuous at $a$, then you can evaluate its limit by finding $f(a)$. That is,

$$ \lim_{x \to a} f(x) = f(a) $$

by the definition of continuity.
Evaluating Limits of Functions at Points of Discontinuity

Rational Functions Containing Holes

When a rational function has the same factor, \((ax - b)\), in its numerator and denominator and all factors in the denominator cancel completely, then its graph will contain a removable discontinuity, or a hole at \(x = \frac{b}{a}\).

For example, consider the function

\[ f(x) = \frac{x - 3}{(x - 3)(x - 1)} \]

and its graph.

Notice that \(f(x)\) is undefined at \(x = 1\) and \(x = 3\).

Simplifying the function, by cancelling the common factors of \((x - 3)\), results in the following expression:

\[ f(x) = \frac{1}{x - 1}, \quad x \neq 1, 3 \]

Evaluating the limit gives

\[ \lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{1}{x - 1} = \frac{1}{3 - 1} = \frac{1}{2} \]

and therefore, the function \(f(x)\) has a hole at the point \((3, \frac{1}{2})\).

Evaluating Limits of Functions at Points of Discontinuity

Rational Functions Containing Holes

To evaluate the limit of rational functions containing holes, begin by factoring the numerator and denominator of the rational function.

Simplify the fraction by completely cancelling all factors common to both the numerator and denominator. After this step, it may be possible to simply substitute the desired value of \(x = a\) into the function. (This will always be the case if the function has a hole at \(x = a\).)
Examples

Example 1

Evaluate \(\lim_{x \to 2} \frac{x^2 + x - 2}{x^2 - 4}\) or show that it does not exist.

Solution

\[
\lim_{x \to 2} \frac{x^2 + x - 2}{x^2 - 4} = \lim_{x \to 2} \frac{(x + 2)(x - 1)}{(x + 2)(x - 2)}
\]

\[
= \lim_{x \to 2} \frac{x - 1}{x - 2}
\]

\[
= \frac{-1}{-2} = \frac{1}{2}
\]

since \(x \neq -2\)

by limit properties 1, 2, 3, and 6

\[
= \frac{3}{4}
\]

Examples

Example 2

Evaluate \(\lim_{x \to 3} \frac{|x - 3|}{x - 3}\)

Solution

Recall that an absolute value function can be written as a piecewise function.

Here,

\[
|x - 3| = \begin{cases} 
  x - 3 & \text{if } x \geq 3 \\
  -x + 3 & \text{if } x < 3 
\end{cases}
\]

Approaching from the right side,

\[
\lim_{x \to 3^-} \frac{|x - 3|}{x - 3} = \lim_{x \to 3^-} \frac{x(x - 3)}{x - 3} = \lim_{x \to 3^-} x = 3
\]

by limit property 2

Approaching from the left side, we have

\[
|x - 3| = -(x - 3) = -x + 3. \text{ Thus,}
\]

\[
\lim_{x \to 3^+} \frac{|x - 3|}{x - 3} = \lim_{x \to 3^+} \frac{x(x - 3)}{x - 3} = \lim_{x \to 3^+} x = \lim_{x \to 3^+} -x = -3
\]

since \(x \neq 3\)

by limit property 2

Since the left and right side limits differ, \(\lim_{x \to 3} \frac{|x - 3|}{x - 3}\) does not exist.
Evaluating Limits of Functions at Points of Discontinuity

Rational Functions Containing Vertical Asymptotes

When a rational function has a factor, \((ax - b)\), in its denominator but not in its numerator, its graph will contain a \textit{vertical asymptote} at \(x = \frac{b}{a}\).

For example, consider the graph of the function \(f(x) = \frac{x - 1}{x + 2}\).

The factor, \(x + 2\), found in the denominator cannot be cancelled with an equal factor in the numerator.

Thus, if we substitute \(x = -2\) into \(f(x)\), the denominator will cause division by 0, which is undefined.

So as \(x \to -2\), the numerator approaches -3, while the denominator approaches 0.

More precisely, as \(x \to -2^+\), \(f(x)\) will be the ratio of a negative number near -3 divided by a small positive number, and \(f(x)\) will approach \(-\infty\) as \(x\) approaches -2 from the right.

On the other hand, as \(x \to -2^-\), \(f(x)\) will be the ratio of a negative number divided by a small negative number, and will thus approach \(+\infty\) as \(x\) approaches -2 from the left.

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Evaluating Limits of Functions at Points of Discontinuity

Rational Functions Containing Vertical Asymptotes

A nice notation for this is to say,

\[
\text{as } x \to -2^+, f(x) \approx \frac{-3}{0^+} \to -\infty, \\
\text{and as } x \to -2^-, f(x) \approx \frac{-3}{0^-} \to +\infty.
\]

The resulting graph is shown.

Therefore, the limit does not exist when \(x \to -2\) because the function is \textit{unbounded} at \(x = -2\).

The limit does exist for all \(x \neq -2\) and can be found by directly substituting the desired value of \(x\) into the function.
Evaluating Limits of Functions at Points of Discontinuity

Rational Functions Containing Vertical Asymptotes

To identify whether the limit of a rational function exists, begin by factoring the numerator and denominator of the rational function.
Simplify by cancelling all equal factors from the numerator and the denominator.
These factors (if completely cancelled) will result in the graph having holes and the limit exists at these holes.

After cancelling, any factor, $ax + b$, that remains in the denominator will result in $f(x)$ having a vertical asymptote
at $x = \frac{b}{a}$.

There are many possibilities for how a function can approach an asymptote.

It may be that
\[ \lim_{x \to \left( \frac{b}{a} \right)^+} f(x) \to -\infty \]

Or it may be that
\[ \lim_{x \to \left( \frac{b}{a} \right)^-} f(x) \to +\infty \]

and
\[ \lim_{x \to \left( \frac{b}{a} \right)^+} f(x) \to +\infty \]

as in the previous example.
In both of these cases, the function's behaviour is different on one side of $x = \frac{b}{a}$ than the other side.

Evaluating Limits of Functions at Points of Discontinuity

Rational Functions Containing Vertical Asymptotes

Alternatively, it may be that
\[ \lim_{x \to \left( \frac{b}{a} \right)^-} f(x) \to +\infty \]

and
\[ \lim_{x \to \left( \frac{b}{a} \right)^+} f(x) \to +\infty \]

This is illustrated in the graph of $f(x) = \frac{1}{(x-1)^2}$.
Note: the limit does not exist, even if the function approaches $\pm\infty$ from the left side and the right side.
Here, the function's behaviour is the same on both sides of the asymptote.

Similarly, we could also have
\[ \lim_{x \to \left( \frac{b}{a} \right)^-} f(x) \to -\infty \]

and
\[ \lim_{x \to \left( \frac{b}{a} \right)^+} f(x) \to -\infty \]
Examples

Example 3

Evaluate \( \lim_{x \to -1} \frac{x^2 - 9}{x^2 + 4x + 3} \), or show the limit does not exist.

Solution

\[
\lim_{x \to -1} \frac{x^2 - 9}{x^2 + 4x + 3} = \lim_{x \to -1} \frac{(x + 3)(x - 3)}{(x + 3)(x + 1)}
\]

\[
= \lim_{x \to -1} \frac{x - 3}{x + 1}, \quad x \neq -3
\]

A hole exists at \( x = -3 \) and a vertical asymptote exists at \( x = -1 \). Hence, the limit does not exist at \( x = -1 \).

Note that it is possible to find \( \lim_{x \to -3} \frac{x^2 - 9}{x^2 + 4x + 3} \). The limit does exist, even though \( x = -3 \) is a hole.

Summary

Consider \( \lim_{x \to a} f(x) \).

The limit of \( f(x) \) as \( x \) approaches \( a \) exists if the function \( f(x) \) approaches the same finite value as \( x \) approaches \( a \) from the right and from the left.

If either one-sided limit does not exist, or they are not equal, then we say that the limit does not exist.

The following methods can be used to evaluate the limit of a rational function.

1. If the function is continuous at \( x = a \), then substitute the desired value, \( a \), into the expression to find \( f(a) \).

   In this case,

   \[
   \lim_{x \to a} f(x) = f(a)
   \]

2. If the rational function is discontinuous at \( x = a \), then \( (x - a) \) is a factor of the denominator.

   Factor the numerator and the denominator in order to cancel any common factors.

   At the end of this step, the factor \( (x - a) \) should not appear in both the numerator and the denominator. (That is, either it does not appear at all or it appears in the numerator or the denominator but not in both.)

   • If every factor of \( (x - a) \) in the denominator is cancelled, then the graph of \( f(x) \) contains a hole at \( x = a \). In this case, we substitute the limit point, \( x = a \), into the simplified expression for \( f(x) \) to obtain the value of the limit.

   • If, after cancellation, at least one factor of \( (x - a) \) remains in the denominator, the graph of \( f(x) \) has a vertical asymptote at \( x = a \). In this case, the limit does not exist.