The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Properties of Definite Integrals

In This Module

We will present some basic properties of definite integrals that will help simplify the process of integration.

Full verifications for most of the properties are beyond the scope of this course, but you will be able to intuitively see why some rules make sense.

For those rules remaining, we will rely on enlightening examples.

Order of Integration Property

In the definition of Riemann sums, we considered an interval $[a, b]$ and hence implicitly assumed that $a < b$.

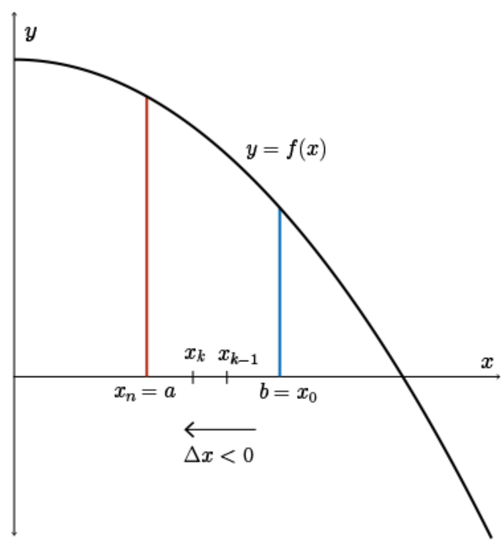
We would like to extend our definition to include the case where $a > b$ or $a = b$.

Given an interval $[a, b]$ (where $a < b$), what happens when we integrate in the reverse direction from b to a?

$$\Delta x = \frac{a-b}{n} = -\frac{b-a}{n}$$

Order of Integration

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$



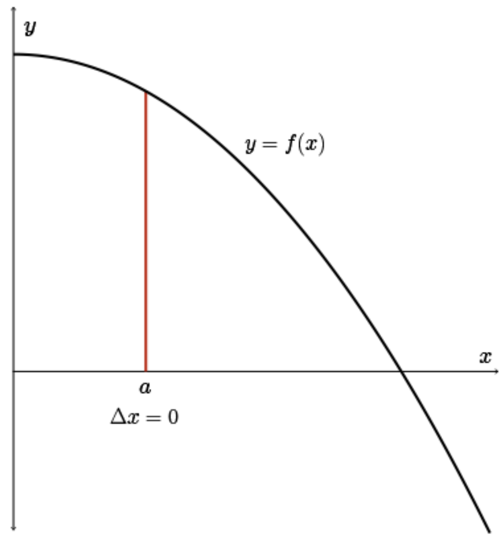
Zero Property

Given $a = b$, what happens when we integrate from a to a?

$$\Delta x = \frac{a-a}{n} = 0$$

Zero Rule

$$\int_a^a f(x)\, dx = 0$$



Additivity Property

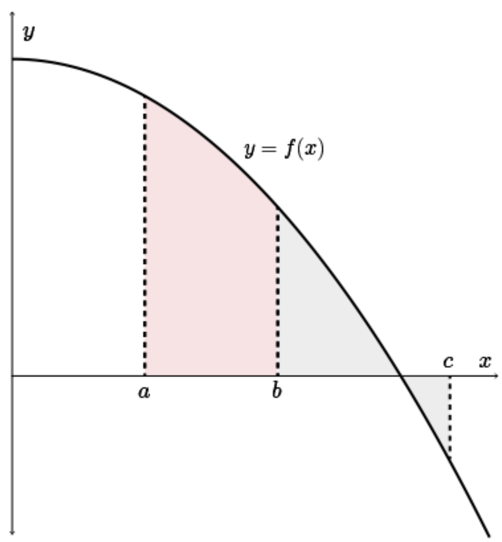
We can combine integrals of the same function over adjacent intervals.

Additivity Rule

$$\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$$

This is harder to prove if we don't have $f(x) \geq 0$ and $a < b < c$.

The result will be the net area bounded by the function over the entire interval $[a, c]$, which is the value of the definite integral from a to c.



Constant Multiples Property

How are the definite integrals $\int_a^b f(x)\,dx$ and $\int_a^b kf(x)\,dx$, where k is any constant, related?

Example 1

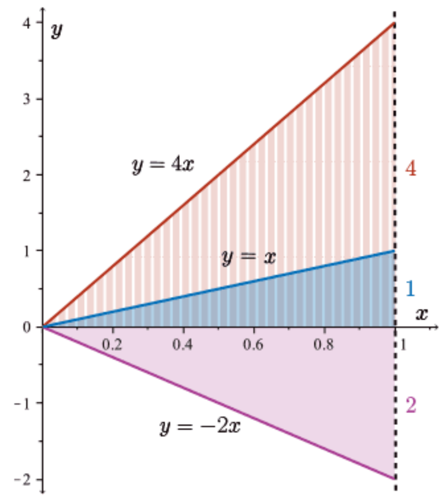
Consider the integrals $\int_0^1 x\,dx$, $\int_0^1 4x\,dx$, and $\int_0^1 -2x\,dx$.

Observations

$$\int_0^1 x\,dx = \frac{1}{2}(1)(1) = \frac{1}{2}$$

$$\int_0^1 4x\,dx = \frac{1}{2}(1)(4) = (4)\left(\frac{1}{2}\right) = 2$$

$$\int_0^1 -2x\,dx = -\frac{1}{2}(1)(2) = (-2)\left(\frac{1}{2}\right) = -1$$



Constant Multiples Property

Example 2

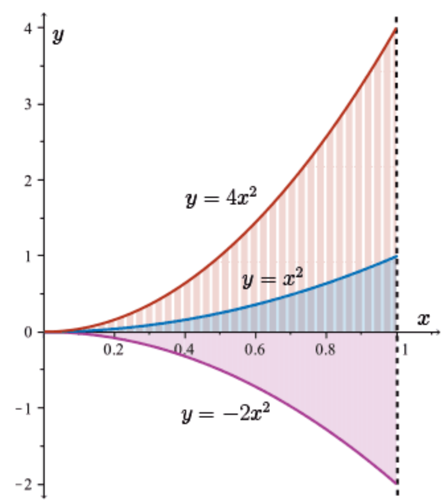
Consider the integrals $\int_0^1 x^2\,dx$, $\int_0^1 4x^2\,dx$, and $\int_0^1 -2x^2\,dx$.

Observations

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

$$\int_0^1 4x^2\,dx = \frac{4}{3} = (4)\left(\frac{1}{3}\right)$$

$$\int_0^1 -2x^2\,dx = -\frac{2}{3} = (-2)\left(\frac{1}{3}\right)$$



Constant Multiples Property

Example 2

Consider the integrals $\int_0^1 x^2\,dx$, $\int_0^1 4x^2\,dx$, and $\int_0^1 -2x^2\,dx$.

Questions

- Can you verify the values of these definite integrals using limits of Riemann sums?
- Can you prove in general that $\int_0^1 kx\,dx = k\int_0^1 x\,dx$ for any number k?
- Can you prove in general that $\int_0^1 kx^2\,dx = k\int_0^1 x^2\,dx$ for any number k?

Constant Multiple Rule

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx \text{ for any } k \in \mathbb{R}$$

Constants (and only constants!) can be "factored out" of an integral.

Sum of Functions Property

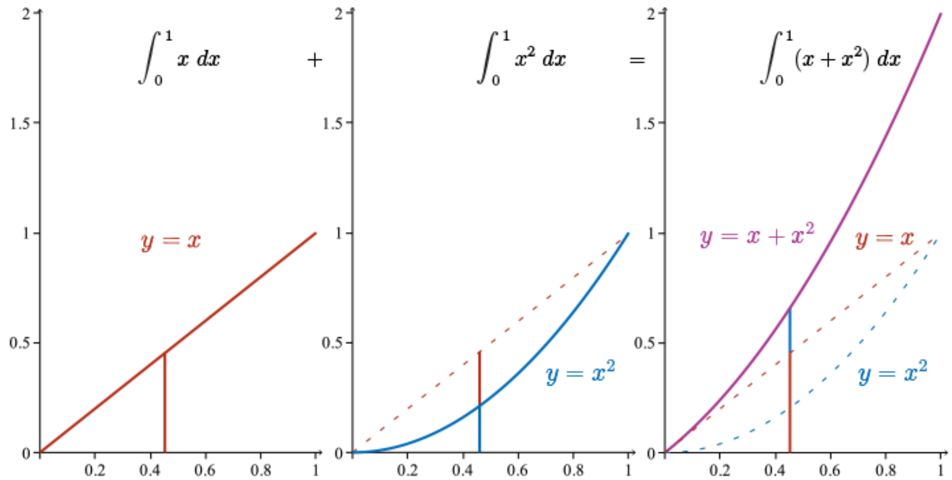
How is the integral $\int_a^b (f(x)+g(x))\,dx$ related to the integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$?

Sum of Functions Property

Example 3

Consider the integrals $\int_0^1 x\,dx$, $\int_0^1 x^2\,dx$, and $\int_0^1 (x+x^2)\,dx$.

Observations



Sum of Functions Property

Sum Rule

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

The integral of a sum is the sum of the integrals.

Difference of Functions Property

How is the integral $\int_a^b (f(x) - g(x))\, dx$ related to the integrals $\int_a^b f(x)\, dx$ and $\int_a^b g(x)\, dx$?

Luckily, we already know how to deal with this case using our rule for constant multiples and our rule for sums.

We can write $f(x) - g(x) = f(x) + (-1) \cdot g(x)$.

Therefore, we have

$$\begin{aligned}
\int_a^b (f(x) - g(x))\, dx &= \int_a^b (f(x) + (-1)g(x))\, dx \\
&= \int_a^b f(x)\, dx + \int_a^b ((-1)g(x))\, dx && \text{sums of functions rule} \\
&= \int_a^b f(x)\, dx + (-1) \int_a^b g(x)\, dx && \text{constant multiples rule } (k = -1) \\
&= \int_a^b f(x)\, dx - \int_a^b g(x)\, dx
\end{aligned}$$

Difference Rule

$$\int_a^b (f(x) - g(x)) dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

The integral of a difference is the difference of the integrals.

Example

Example 4

Given that $\int_{-1}^1 f(x)\, dx = 3$, $\int_1^2 f(x)\, dx = -4$, and $\int_{-1}^1 g(x)\, dx = -2$, evaluate the following definite integrals:

Solution

1. $\int_{-1}^1 \frac{1}{2} f(x)\, dx = \underbrace{\frac{1}{2} \int_{-1}^1 f(x)\, dx}_{\text{constant multiple}}$

   $= \frac{1}{2}(3) = \frac{3}{2}$

2. $\int_1^1 g(x)\, dx = 0$ by the zero rule

Example

Example 4

Given that $\int_{-1}^{1} f(x)\,dx = 3$, $\int_{1}^{2} f(x)\,dx = -4$, and $\int_{-1}^{1} g(x)\,dx = -2$, evaluate the following definite integrals:

Solution

3. $$\int_{-1}^{1}(f(x) - 3g(x))\,dx = \underbrace{\int_{-1}^{1} f(x)\,dx - \int_{-1}^{1} 3g(x)\,dx}_{\text{difference of functions}}$$
$$= \int_{-1}^{1} f(x)\,dx - \underbrace{3\int_{-1}^{1} g(x)\,dx}_{\text{constant multiple}}$$
$$= 3 - 3(-2) = 9$$

Example

Example 4

Given that $\int_{-1}^{1} f(x)\,dx = 3$, $\int_{1}^{2} f(x)\,dx = -4$, and $\int_{-1}^{1} g(x)\,dx = -2$, evaluate the following definite integrals:

Solution

4. $$\int_{-1}^{2} 2f(x)\,dx = \underbrace{2\int_{-1}^{2} f(x)\,dx}_{\text{constant multiple}}$$
$$= 2\left[\underbrace{\int_{-1}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx}_{\text{additivity}}\right]$$
$$= 2[3 + (-4)] = -2$$

Example

Example 4

Given that $\int_{-1}^{1} f(x)\,dx = 3$, $\int_{1}^{2} f(x)\,dx = -4$, and $\int_{-1}^{1} g(x)\,dx = -2$, evaluate the following definite integrals:

Solution

5. $$\int_{1}^{-1} (-4f(x) + 5g(x))\,dx = \underbrace{\int_{1}^{-1} -4f(x)\,dx + \int_{1}^{-1} 5g(x)\,dx}_{\text{sum of functions}}$$

$$= \underbrace{-4\int_{1}^{-1} f(x)\,dx}_{\text{constant multiple}} + \underbrace{5\int_{1}^{-1} g(x)\,dx}_{\text{constant multiple}}$$

$$= \left[+4 \underbrace{\int_{-1}^{1} f(x)\,dx}_{\text{order of integration}}\right] + \left[-5 \underbrace{\int_{-1}^{1} g(x)\,dx}_{\text{order of integration}}\right]$$

$$= 4(3) + (-5)(-2) = 22$$

Summary of the Properties of Definite Integrals

1. Zero: $\int_{a}^{a} f(x)\,dx = 0$
2. Order of Integration: $\int_{b}^{a} f(x)\,dx = -\int_{a}^{b} f(x)\,dx$
3. Additivity: $\int_{a}^{b} f(x)\,dx + \int_{b}^{c} f(x)\,dx = \int_{a}^{c} f(x)\,dx$
4. Constant Multiples: $\int_{a}^{b} kf(x)\,dx = k\int_{a}^{b} f(x)\,dx$ for any $k \in \mathbb{R}$
5. Sum: $\int_{a}^{b} (f(x) + g(x))\,dx = \int_{a}^{b} f(x)\,dx + \int_{a}^{b} g(x)\,dx$
6. Difference: $\int_{a}^{b} (f(x) - g(x))\,dx = \int_{a}^{b} f(x)\,dx - \int_{a}^{b} g(x)\,dx$

Remarks:

1. These rules only apply if all of the integrals in question are defined.
2. It is important to note that, although our explanations and pictures dealt mainly with positive functions and $a < b < c$, each of these properties are true for **any** continuous functions $f(x), g(x)$ and **any** numbers $k, a, b,$ and c.

Summary of the Properties of Definite Integrals

The Variable of Integration

The choice of the variable x in the integral $\int_a^b f(x)\,dx$ is not important.

In other words, the integrals

$$\int_a^b f(x)\,dx,\ \int_a^b f(y)\,dy,\ \int_a^b f(t)\,dt,\ \int_a^b f(u)\,du$$

are all different ways of writing the *same* quantity.

Each definite integral represents the computation of the area bounded by the function f from a to b.

The function, f, and the endpoints, a and b, remain the same; only the variable of integration is changing.

Challenge Example

In the next module, we will discuss the fundamental theorem of calculus, which connects the two branches: differential calculus and integral calculus.

Here, we discuss an example to help motivate this result due to Newton and Leibniz.

Try the following exercise:

Pick your favourite continuous function $f(x)$ over some closed interval $[a, b]$.

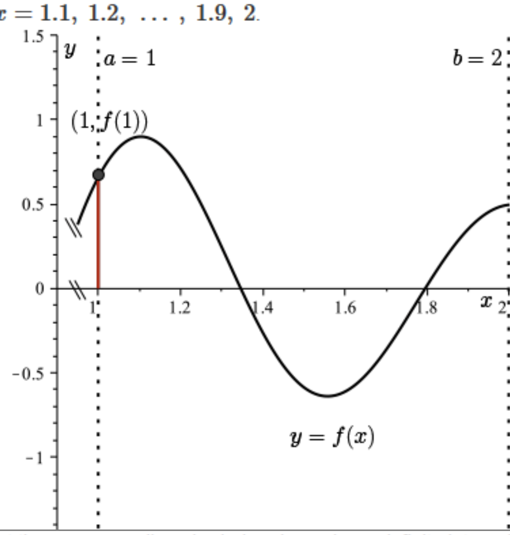
For this example, we will use the function $f(x) = \frac{1}{x}\sin(7x)$ over $[1, 2]$, but the choice doesn't really matter!

Choose any x value satisfying $1 \leq x \leq 2$.

Draw a vertical line at $x = a$ and at your value of x and colour in the region bounded by $f(x)$, the x-axis and your two vertical lines.

Observe that if you select a different x value, then the shaded region will be different.

Here are some examples for $x = 1.1,\ 1.2,\ \ldots\ ,\ 1.9,\ 2$.



For a fixed x, we can represent the corresponding shaded region using a definite integral.

Here we need to be a bit careful with notation.

We are using x twice, once to represent the point between 1 and 2 that we fix, and once to represent the variable in the function $f(x)$.

To get around this, we will think of f being a function of a "new" variable t.

So we have the function $f(t) = \frac{1}{t}\sin(7t)$ from $t = 1$ to $t = 2$.

Now if we fix $1 \leq x \leq 2$, the corresponding shaded region is the region bounded by the curve $y = f(t)$ and the x-axis from $t = 1$ to $t = x$.

In other words, the shaded region is the value of the definite integral

$$\int_1^x f(t)\, dt$$

But, this definite integral is a function of x!

Let's call this function F.

Then, for each $1 \leq x \leq 2$ we have

$$F(x) = \int_1^x f(t)\, dt = \textit{Net area of the shaded region from } t = 1 \textit{ to } t = x$$

Question: What is the derivative of the function $F(x)$?

Recall the formal definition of the derivative

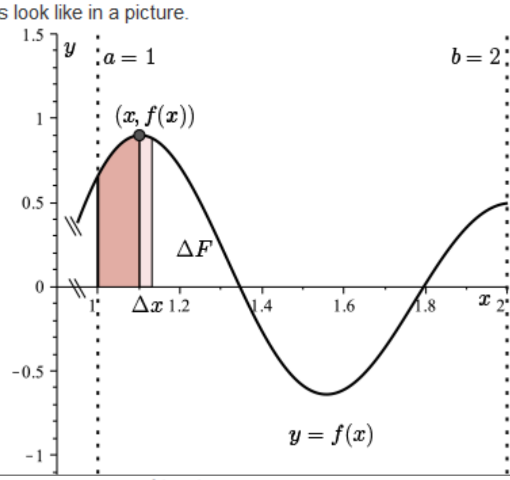
$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

The denominator of this fraction is the horizontal change in x, which we will denote by Δx; the numerator is the corresponding change in the function F from x to $x + h$, which we will denote by ΔF.

In this new notation our equation becomes

$$F'(x) = \lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x}$$

Let's see what these quantities look like in a picture.



This is a graph to help represent the value of $F'(1.1)$, but the same method works for $F'(x)$ for any x in the interval $(1, 2)$.

We have vertical lines at $x = 1$ and $x = 1.1$, the value $F(1.1)$ is the area of the dark shaded region (from $x = 1$ to $x = 1.1$), the value ΔF is the area of the light shaded strip, and the value Δx is the base of the light shaded strip.

What is happening to the quantity $\dfrac{\Delta F}{\Delta x}$ as $\Delta x \to 0$?

If Δx is *small enough* then the light shaded region, ΔF, is essentially the area of a rectangle.

$$\Delta F \approx base \times height \approx \Delta x \times f(x)$$

We conclude that $\dfrac{\Delta F}{\Delta x} \approx f(x)$ for *small* Δx, and so our guess is that

$$\lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} = f(x)$$

Let's see what is happening numerically with our example.

Δx	ΔF	$\Delta F/\Delta x$
0.05	$\int_{1.1}^{1.15} f(x)\,dx = 0.4421662986$	0.8843325972
0.04	$\int_{1.1}^{1.14} f(x)\,dx = 0.03559761280$	0.8899403200
0.03	$\int_{1.1}^{1.3} f(x)\,dx = 0.02682520321$	0.8941734402
0.02	$\int_{1.1}^{1.12} f(x)\,dx = 0.01794002380$	0.8970011900
0.01	$\int_{1.1}^{1.11} f(x)\,dx = 0.008983962260$	0.8983962260

Note that $f(1.1) = 0.8983347581$.

It looks like $\dfrac{\Delta F}{\Delta x} \to f(1.1)$ as $\Delta x \to 0$.

Do you believe that $F'(x) = \lim\limits_{\Delta x \to 0} \dfrac{\Delta F}{\Delta x} = f(x)$?

In other words, do you believe that $F(x) = \displaystyle\int_1^x f(t)\,dt$ is an antiderivative of $f(x)$?