The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

Sigma Notation and Riemann Sums

In This Module

- We will introduce sigma notation — a compact way of writing large sums of like terms — and define the notion of a Riemann sum.
- We will see that Riemann sums are a generalization of the rectangular approximations that we saw in the previous module, and they will be used to estimate the area of regions in the plane.

As usual, all of our regions will be bounded by continuous functions, but in this section, we will work with functions that may lie both above and below the x-axis, so we will introduce the idea of "net area."

Sigma Notation

Consider the sum of the first 7 positive integers $1+2+3+4+5+6+7$.
As the number 7 is quite small, it is not too time consuming to write this entire sum out by hand.
But how do we write the sum of the first 100 positive integers?

$$1+2+\cdots+99+100$$

Sigma notation can be used to express a sum of the form $a_1+a_2+\cdots+a_{n-1}+a_n$ compactly as

$$a_1+a_2+\cdots+a_{n-1}+a_n=\sum_{k=1}^{n}a_k$$

The capital Greek letter Σ (sigma) stands for "sum" and k is called the index of summation.
The number at the bottom of the Σ symbol tells us where to start our sum (in this case at $k=1$) and the number at the top tells us where to end our sum (in this case at $k=n$).

How do we use this to write the sum of the first 100 positive integers $1+2+3+\cdots+99+100$?
The first term in our sum is 1, the second term is 2, the third term is 3, so in general, the "k^{th}" term in our sum is k.

From this, we see $a_k=k$ and that our sum goes from $k=1$ to $k=100$.
In sigma notation, we get that

$$1+2+3+\cdots+99+100=\sum_{k=1}^{100}k$$

Examples

Example 1

Find the value of the sum $\sum_{k=1}^{5}(2k-1)$.

Solution

This notation tells us to sum terms of the form $a_k = 2k-1$ from $k=1$ to $k=5$.
So we have

$$\begin{aligned}\sum_{k=1}^{5}(2k-1) &= (2(1)-1)+(2(2)-1)+(2(3)-1)+(2(4)-1)+(2(5)-1)\\ &= 1+3+5+7+9\\ &= 25\end{aligned}$$

This is a short form for taking the sum of the first 5 odd positive integers.

All odd positive integers can be expressed as $2k-1$ for some positive integer k, so we have a compact way of writing sums of consecutive odd positive integers.

How do we write the sum of the first n odd positive integers using sigma notation?

Examples

Example 2

Write the sum of the first 6 positive even numbers using sigma notation.

Solution

Following the lead of example 1, we note that every even positive integer can be written as $2k$ for some positive integer k.
So we want $a_k = 2k$ in this case.
We need the first 6 such numbers so we take our sum from $k=1$ to $k=6$, so we want the sum

$$\sum_{k=1}^{6} 2k$$

We can check that

$$\begin{aligned}\sum_{k=1}^{6} 2k &= 2(1)+2(2)+2(3)+2(4)+2(5)+2(6)\\ &= 2+4+6+8+10+12\end{aligned}$$

This is, indeed, the correct sum.

Remarks

1. To represent the sum of the first n positive even numbers, we simply need to change our end point in example 2 from $k = 6$ to the more general $k = n$.

In sigma notation we get

$$\sum_{k=1}^{n} 2k = 2(1) + 2(2) + \cdots + 2(n-1) + 2(n)$$

2. We now have a compact way of writing the formulas for the sums $1 + 2 + \cdots + n$ and $1^2 + 2^2 + \cdots + n^2$, which will be useful in the coming modules.

In sigma notation, we have

$$\sum_{k=1}^{n} k = \frac{n(n-1)}{2} \qquad \text{and} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

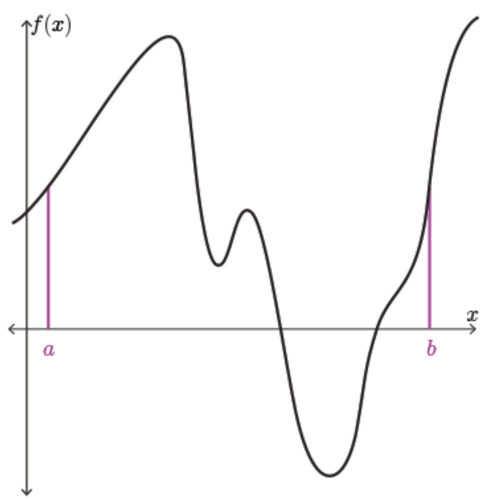
Riemann Sums

Let $f(x)$ be a continuous function on a closed interval $[a, b]$.

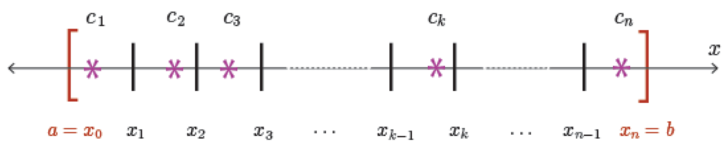
Informally speaking, a **Riemann sum** of f on the interval $[a, b]$ is formed by taking the sum of the areas of finitely many rectangles, whose heights are all determined by values of $f(x)$ over the interval.

This should sound reminiscent of the rectangular approximation from the last section.

Here is the (more general) method for producing a Riemann sum.



Riemann Sums



Step 1

Fix a positive integer n and divide the interval $[a, b]$ into n subintervals of equal length.
In other words, find the $n - 1$ points between a and b satisfying $a < x_1 < x_2 < \ldots < x_{n-1} < b$ and such that

$$x_1 - a = x_2 - x_1 = \cdots = x_k - x_{k-1} = \cdots = b - x_{n-1}$$

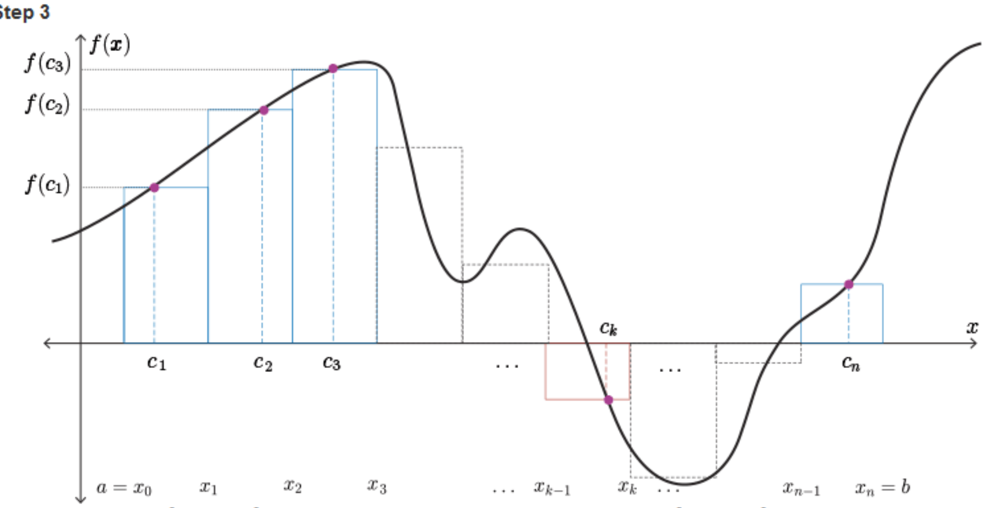
We will denote this constant difference Δx.
Let $x_0 = a$ and $x_n = b$. Then, the set $P = [x_0, x_1, \ldots, x_{n-1}, x_n]$ defines what we call a regular partition of the interval $[a, b]$.
As the length of the entire interval is $b - a$, we have $\Delta x = \dfrac{b-a}{n}$.

Step 2

Choose a single point in each of the n subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$.
The point chosen in the interval $[x_{k-1}, x_k]$ will be labelled c_k. We will call $c_1, \ldots, c_n$, the sample points.

Riemann Sums

Step 3



On each subinterval $[x_{k-1}, x_k]$, we form a rectangle whose base is the interval $[x_{k-1}, x_k]$ and whose height is $|f(c_k)|$.
We do so in such a way that the rectangle touches the curve at the point $(c_k, f(c_k))$.
Note that the value of $f(c_k)$ may be negative!
In this case, we draw our rectangle below the x-axis.

Riemann Sums

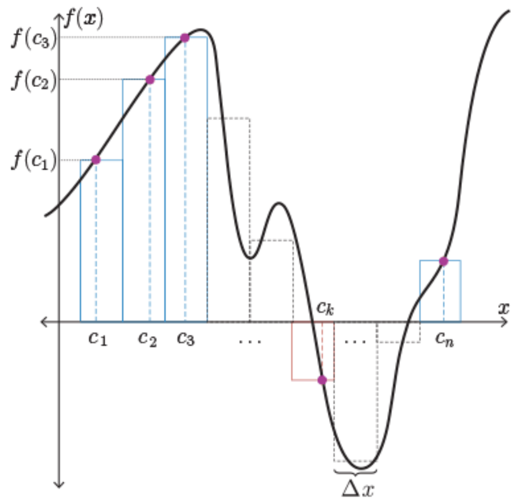
Step 4

Construct the sum of the "signed areas" of the n rectangles.

$$f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + \cdots + f(c_k)\Delta x + \cdots + f(c_n)\Delta x = \sum_{k=1}^{n} f(c_k)\Delta x$$

This finite sum is called a Riemann sum for f on the interval $[a, b]$.

- The quantity $|f(c_k)|\Delta x$ is the area of the k^{th} rectangle.
- If the function value $f(c_k)$ is a *positive* number, then the quantity $f(c_k)\Delta x$ in our sum is *positive*. This tells us that we are *adding* the area of the k^{th} rectangle to our running tally.
- If the function value $f(c_k)$ is a *negative* number, then the quantity $f(c_k)\Delta x$ in our sum is *negative*. This tells us that we are *subtracting* the area of the k^{th} rectangle from our running tally.



Riemann Sums

Our Riemann sum is the "net area" (sometimes called the "signed area") of the n rectangles.

$$\sum_{k=1}^{n} f(c_k)\Delta x = (\text{area of rectangles lying above the } x\text{-axis}) - (\text{area of rectangles lying below the } x\text{-axis})$$

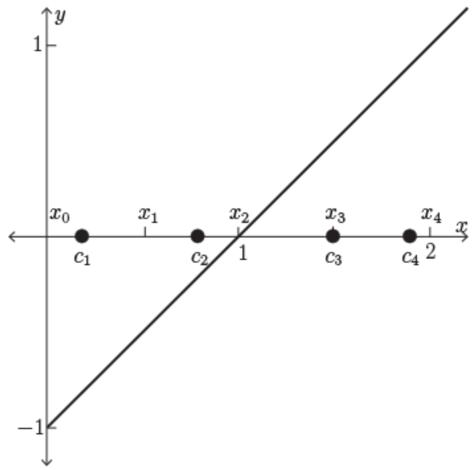
Each Riemann sum is a real number, and a Riemann sum with n subintervals can be thought of as an approximation of the "net area" between the curve and the x-axis over the interval $[a, b]$ using n (signed) rectangles.
As n gets larger, we get more rectangles in our Riemann sum, and the rectangles become thinner.

This suggests that a larger value of n should correspond to a "better" approximation to the net area.

Examples

Example 3

a. Calculate a Riemann sum for $f(x) = x - 1$ over $[0, 2]$ using $n = 4$.



Solution

Step 1: Divide the interval $[0, 2]$ into 4 subintervals of length $\Delta x = \frac{2-0}{4} = \frac{1}{2}$.

So we have a regular partition, $P = [x_0, x_1, x_2, x_3, x_4]$ of $[0, 2]$, where

$$x_0 = 0, \;\; x_1 = \frac{1}{2}, \;\; x_2 = 1, \;\; x_3 = \frac{3}{2}, \;\; \text{and} \;\; x_4 = 2$$

Step 2: Choose a sample point, c_k, in each of the 4 subintervals $\left[0, \frac{1}{2}\right]$, $\left[\frac{1}{2}, 1\right]$, $\left[1, \frac{3}{2}\right]$, and $\left[\frac{1}{2}, 2\right]$.

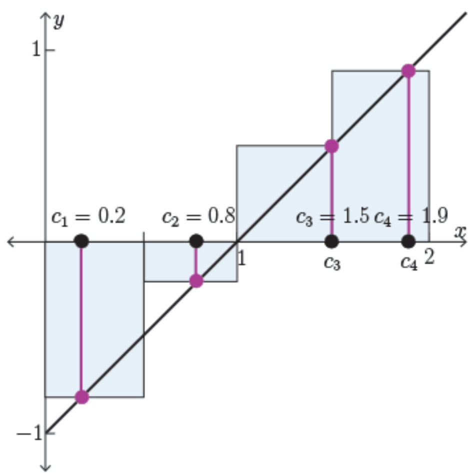
We have chosen

$$\begin{aligned} c_1 &= 0.2 \\ c_2 &= 0.8 \\ c_3 &= 1.5 = x_3 \\ c_4 &= 1.9 \end{aligned}$$

Examples

Example 3

a. Calculate a Riemann sum for $f(x) = x - 1$ over $[0, 2]$ using $n = 4$.



Solution

Step 3: Form 4 rectangles with bases $\left[0, \frac{1}{2}\right]$, $\left[\frac{1}{2}, 1\right]$, $\left[1, \frac{3}{2}\right]$, and $\left[\frac{1}{2}, 2\right]$ and heights determined by $f(c_1)$, $f(c_2)$, $f(c_3)$, and $f(c_4)$:

$$\begin{aligned} f(c_1) &= f(0.2) = 0.2 - 1 = -0.8 \\ f(c_2) &= f(0.8) = 0.8 - 1 = -0.2 \\ f(c_3) &= f(1.5) = 1.5 - 1 = 0.5 \\ f(c_4) &= f(1.9) = 1.9 - 1 = 0.9 \end{aligned}$$

Step 4: We form the Riemann sum

$$\sum_{k=1}^{4} f(c_k)\Delta x = f(c_1)\left(\frac{1}{2}\right) + f(c_2)\left(\frac{1}{2}\right) + f(c_3)\left(\frac{1}{2}\right) + f(c_4)\left(\frac{1}{2}\right)$$

$$= \frac{-0.8}{2} + \frac{-0.2}{2} + \frac{0.5}{2} + \frac{0.9}{2} = 0.2$$

Examples

Example 3

b. Calculate a Riemann sum for $f(x) = x - 1$ over $[0, 2]$ using $n = 6$ and taking the sample points c_k to be the right endpoints of the subintervals.

Solution

Step 1 — Partition: $\Delta x = \dfrac{2-0}{6} = \dfrac{1}{3}$, so $x_0 = 0$, $x_1 = \dfrac{1}{3}$, $x_2 = \dfrac{2}{3}$, $x_3 = 1$, $x_4 = \dfrac{4}{3}$, $x_5 = \dfrac{5}{3}$, and $x_6 = 2$.

Step 2 — Sample points: c_k = right endpoints, so $c_1 = x_1 = \dfrac{1}{3}$, $c_2 = x_2 = \dfrac{2}{3}$, . . . , and in general

$$c_k = x_k = \frac{k}{3}$$

Steps 3 and 4 — Riemann sum:

$$\begin{aligned}\sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{6} f(x_k)\Delta x \\ &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\ &= \Delta x\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) + f\left(\frac{3}{3}\right) + f\left(\frac{4}{3}\right) + f\left(\frac{5}{3}\right) + f\left(\frac{6}{3}\right)\right] \\ &= \frac{1}{3}\left[-\frac{2}{3} - \frac{1}{3} + 0 + \frac{1}{3} + \frac{2}{3} + 1\right] = \frac{1}{3}\end{aligned}$$

Comments

1. When calculating Riemann sums, we can skip the pictures and skip step 3 entirely; however, while we are learning, we will generally include the pictures to help us keep track of what these Riemann sums are really "doing."
Can you use the graph of $f(x) = x - 1$ over $[0, 2]$ and the "net area" interpretation of Riemann sums to determine what these Riemann sums are approximating?
Which approximation is "better"?

2. In integral calculus, we are mostly interested in the limits of the Riemann sums as n (the number of subintervals) goes to infinity.
We will see in the next module that, for continuous functions, the choice of sample points doesn't matter as n gets large.
As such, we will generally pick the sample points uniformly as left endpoints, right endpoints, or midpoints of the subinterval.