The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

The Fundamental Theorem of Calculus

In This Module

- We will discuss the fundamental theorem of calculus, a theorem connecting the process of integration with the process of differentiation.
- As a result of this theorem, we will gain a powerful tool for evaluating definite integrals very easily, without considering Riemann sums or even net areas.

The Fundamental Theorem, Part 1

As we have seen, differential calculus is the calculus for finding slopes of tangent lines while integral calculus is the calculus for finding areas under curves.

When stated in this theoretical way, these two branches of calculus do not seem to have any obvious connection; however, if you recall the physical interpretations of these two problems, you may suspect that there is indeed a link. Differential calculus was motivated as a means to solve problems about instantaneous velocities and integral calculus was introduced as a method for calculating distances.

As we know, these two problems are fundamentally linked since the velocity function is the derivative of the position function.

Let's revisit the final problem from the previous module.

Given a function $f(t)$, continuous over some interval $[a, b]$, consider the function $F(x)$ defined by

$$F(x) = \int_a^x f(t)\,dt,\ a \leq x \leq b$$

The value $F(x)$ is the net area between the curve $y = f(t)$ and the t-axis from $t = a$ to $t = x$, so F is indeed a function of the variable x.

If we fix a number $x = c$ in the interval $[a, b]$, then we have $F(c) = \int_a^c f(t)\,dt$, which is a real number equal to the net area from $t = a$ to $t = c$.

The first part of the fundamental theorem states that $F'(x) = f(x)$.

The Fundamental Theorem, Part 1

The Fundamental Theorem of Calculus, Part 1

If f is continuous on the interval $[a, b]$, then the function defined by

$$F(x) = \int_a^x f(t)\,dt,\ a \le x \le b$$

is continuous on $[a, b]$, differentiable on (a, b), and $F'(x) = f(x)$.

Remarks

1. We call our function "F" here to match the symbol we used when we introduced antiderivatives. This is because our function $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of $f(x)$.
2. This theorem asserts that every continuous function has an antiderivative.
3. We can rewrite the statement $F'(x) = f(x)$ using Leibniz notation to get the following:
$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$
This shows the connection between integration and differentiation.
4. We will omit a proof of the theorem, but the final problem from the previous module provides some intuition for why this theorem is true. We will instead explore some examples.

Examples

Example 1

Consider the function $f(x) = x$ over the interval $[0, 10]$.

Let $F(x) = \int_0^x f(t)\,dt = \int_0^x t\,dt$ be defined for $0 \le x \le 10$.

Here, we can actually get a nice formula for the function $F(x)$. Recall that for any interval $[a, b]$ we have the formula

$$\int_a^b t\,dt = \frac{b^2 - a^2}{2}$$

If we let $a = 0$ and $b = x$, then we have $\int_0^x t\,dt = \frac{x^2 - (0)^2}{2} = \frac{x^2}{2}$ and hence $F(x) = \int_0^x t\,dt = \frac{x^2}{2}$.

This function is indeed continuous on $[0, 10]$ and differentiable on $(0, 10)$ and

$$F'(x) = \frac{1}{2}(2x) = x = f(x)$$

Remark

Notice that the function $F(x) = \frac{x^2}{2}$ is actually continuous and differentiable everywhere, not just on the intervals $[0, 10]$ and $(0, 10)$ respectively.

The statement of the theorem should be interpreted as "If f is continuous on $[a, b]$, then F is guaranteed to also be continuous on $[a, b]$ and differentiable on (a, b)."

Examples

Example 2

Find the derivative of the functions $g(x) = \int_2^x \left(t^2 + \sin(t)\right)\, dt$ and $h(x) = \int_2^{x^2} \left(t^2 + \sin(t)\right)\, dt$.

Recall: If $F(x) = \int_a^x f(t)\, dt$, then $F'(x) = f(x)$.

Solution

Let $f(t) = t^2 + \sin(t)$.
Then, $f(t)$ is continuous over any interval $[a, b]$ and $g(x)$ is in the exact form of part 1 of the fundamental theorem:
$g(x) = \int_2^x f(t)\, dt$ with $a = 2$.
Therefore,

$$g'(x) = f(x) = x^2 + \sin(x)$$

The function $h(x)$ is not in the correct form to apply our theorem as the upper limit of integration is x^2, not x.

We need to rewrite our integral to match the theorem and to do so, we will use a change of variables with $u = x^2$.

Now, we can rewrite our function as $h(u) = \int_2^u \left(t^2 + \sin(t)\right)\, dt$, remembering that u is a function of x.

Examples

Example 2

Find the derivative of the functions $g(x) = \int_2^x \left(t^2 + \sin(t)\right)\, dt$ and $h(x) = \int_2^{x^2} \left(t^2 + \sin(t)\right)\, dt$.

Recall: If $F(x) = \int_a^x f(t)\, dt$, then $F'(x) = f(x)$.

Solution

To differentiate $h(x)$, we will need to use the chain rule along with the fundamental theorem.

We have $h(u(x))$. By the chain rule $\frac{d}{dx}\left(h(u(x))\right) = \frac{d}{du}\left(h(u)\right) \cdot \frac{d}{dx}\left(u(x)\right)$, so

$$\begin{aligned} \frac{d}{dx}\left(\int_2^{x^2} \left(t^2 + \sin(t)\right)\, dt\right) &= \frac{d}{du}\left[\int_2^u \left(t^2 + \sin(t)\right) dt\right] \cdot \frac{d}{dx}\left(x^2\right) \\ &= \underbrace{\left(u^2 + \sin(u)\right)}_{\text{by part 1}}(2x) \\ &= \left((x^2)^2 + \sin(x^2)\right)(2x) \\ &= 2x^5 + 2x\sin(x^2) \end{aligned}$$

The Fundamental Theorem, Part 2

The second part of the fundamental theorem of calculus follows quite quickly from the first part.

Informally speaking, it states that if we know an antiderivative F of a function f, then we can evaluate a definite integral of f using only information about F without using limits of Riemann sums or considering net areas.

The Fundamental Theorem of Calculus, Part 2

If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

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If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

Proof

Let $G(x) = \int_a^x f(t)\,dt$.

From the fundamental theorem of calculus, part 1, we have $G'(x) = f(x)$.

That is, G is an antiderivative of f.

The Fundamental Theorem, Part 2

The Fundamental Theorem of Calculus, Part 2

If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

Proof

Recall that if $F(x)$ is any other antiderivative of $f(x)$, then $F(x) = G(x) + c$ for some constant c.

Let's start with the right hand side of the equality in the theorem and work our way to the expression on the left hand side.

$$\begin{aligned} F(b) - F(a) &= \Big[G(b) + c\Big] - \Big[G(a) + c\Big] && \text{as } F = G + c \\ &= G(b) - G(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt && \text{by the definition of } G \\ &= \int_a^b f(t)\,dt - 0 && \text{by the zero rule} \\ &= \int_a^b f(t)\,dt \end{aligned}$$

The Fundamental Theorem, Part 2

Remarks

1. The proof of part 1 (which was omitted from the lesson) requires most of the effort and contains the main observation connecting the two branches of calculus: the area function $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of f. Most applications of the fundamental theorem will use part 2.
2. Using part 2, we can now evaluate definite integrals without computing rectangular approximations, without limits, and often using only one line of computation. All that we require is the ability to find an antiderivative of the integrand f.

Examples

Example 3

Recall that if the position of a particle at time t is given by the function $s(t)$, then the velocity of the particle is $v(t) = s'(t)$.

In other words, the position function is an antiderivative of the velocity function.

So by part 2 of the fundamental theorem,

$$\int_a^b v(t)\,dt = s(b) - s(a)$$

Assuming we have $v(t) \geq 0$ over the interval $[a, b]$, the left hand side of the equation is the area under the velocity curve over $[a, b]$ and the right hand side is the total distance traveled.

This is a formal justification of what we intuitively believed to be true when we first began our discussion about finding distances traveled given a varying (positive) velocity.

We have a similar physical interpretation if $v(t)$ also takes on negative values over the interval, but we will leave that discussion until the next unit.

Examples

Example 4

Evaluate the following integrals using the fundamental theorem, part 2.

a. $\int_0^1 x^2\,dx$ **b.** $\int_1^2 e^x\,dx$ **c.** $\int_a^b x\,dx$

Solution

a. From part 2 of the fundamental theorem, we have that $\int_0^1 x^2\,dx = F(1) - F(0)$, where F is any antiderivative of $f(x) = x^2$.

Recall, from the module on antiderivatives, that one antiderivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$.

We have $F(1) = \frac{1}{3}(1)^3 = \frac{1}{3}$ and $F(0) = 0$ and therefore,

$$\int_0^1 x^2 dx = F(1) - F(0) = \frac{1}{3} - 0 = \frac{1}{3}$$

Now, go back to Computing Areas where you were originally asked to find this area using limits of rectangular approximations and compare the solutions.

Examples

Example 4

Evaluate the following integrals using the fundamental theorem, part 2.

a. $\int_0^1 x^2\,dx$ **b.** $\int_1^2 e^x\,dx$ **c.** $\int_a^b x\,dx$

Notation

We often use the notation $\Big[F(x)\Big]_a^b$ or $F(x)\Big|_a^b$ to denote $F(b) - F(a)$. This will allow us to organize our solutions onto one line.

To be consistent, we will use the notation $\Big[F(x)\Big]_a^b$ throughout the unit.

For example, we will write $\int_0^1 x^2\,dx = \left[\frac{1}{3}\,x^3\right]_0^1 = \frac{1}{3}\,(1)^3 - \frac{1}{3}\,(0)^3 = \frac{1}{3}$.

Examples

Example 4

Evaluate the following integrals using the fundamental theorem, part 2.

a. $\int_0^1 x^2\,dx$ **b.** $\int_1^2 e^x\,dx$ **c.** $\int_a^b x\,dx$

Solution

b. Since $\frac{d}{dx}\left(e^x\right) = e^x$, an antiderivative of $f(x) = e^x$ is $F(x) = e^x$. By part 2, we have

$$\int_1^2 e^x\,dx = \Big[e^x\Big]_1^2 = e^2 - e^1 = e^2 - e$$

If you remember, this was our challenge example at the end of Sigma Notation and Riemann Sums. Our first solution involved Riemann sums, limits, geometric series and l'Hospital's rule.

c. This is our third time seeing this example in different contexts: as a limit of Riemann sums, as a net area between a curve and the x-axis, and now using the fundamental theorem.

As $\frac{1}{2}\,x^2$ is an antiderivative of x, we have

$$\int_a^b x\,dx = \left[\frac{1}{2}\,x^2\right]_a^b = \frac{1}{2}\,b^2 - \frac{1}{2}\,a^2 = \frac{b^2 - a^2}{2}$$

Conclusion

This completes our introduction to definite integrals.

In summary, we have three interpretations of the definite integral:

1. The definite integral as a limit of Riemann sums (definition).
2. The definite integral as a net area.
3. The definite integral as the difference of an antiderivative at the upper limit of integration and the lower limit of integration.