The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Linearization and Newton's Method

In This Module

- We will learn Newton's method for approximating the roots of an equation of the form $f(x) = 0$.

The quadratic formula will find the exact roots in the case where $f(x)$ is a quadratic polynomial, but how about the roots of a function like $f(x) = \sin(x) - x^2$?
The fundamental tool used in this method is the tangent line approximation.

Finding Roots

Question

What are the roots of the equation $ax^2 + bx + c = 0$?

Solution

We have a well-known quadratic formula that finds the roots of any quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

There are also formulas for finding the roots of equations involving 3^{rd} and 4^{th}-degree polynomials, although they are much more complicated than the above.
For polynomials that are degree 5 and above, there is no such formula.
How might we approximate the roots of such an equation? What if our equation involves functions other than polynomials?

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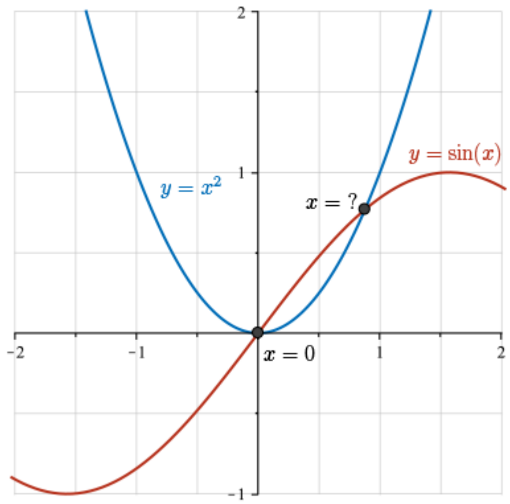
Finding Roots

Question

What are the roots of the equation $\sin(x) - x^2 = 0$?
If you graph the curves $y = \sin(x)$ and $y = x^2$, then you will see that these functions intersect at exactly 2 points, and hence there are exactly 2 roots of the above equation.
One of these roots is $x = 0$ but what is the value of the other root?
We cannot get an exact answer for the 2^{nd} root of this equation, but we can approximate the root to any degree of accuracy that we would like.
One method to do so is called Newton's method (also called the Newton-Raphson method) and is a relatively straightforward application of tangent line approximation.



Linearization

If there is a tangent line to the curve $y = f(x)$ at the point $(a, f(a))$, then this tangent line may provide a good approximation of the curve near $x = a$.
That is, since the curve lies close to this tangent line near the point $(a, f(a))$, this line can be used to approximate the values of the function.

Recall that the equation of the line having slope m and passing through the point (x_1, y_1) is given by

$$y - y_1 = m(x - x_1)$$

The tangent line to $y = f(x)$ at the point $x = a$ has slope $f'(a)$ and passes through the point $(a, f(a))$, and hence the equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

If $f(a)$ and $f'(a)$ can be readily calculated, then this tangent line can be useful for approximating values of $f(x)$ for x near a.

The linearization of f at a is defined to be the function

$$L(x) = f(a) + f'(a)(x - a)$$

Then, we have $f(x) \approx L(x)$ if $x \approx a$. This approximation is called the linear approximation of $f(x)$ at $x = a$.

Using Linearization to Approximate Square Roots

Example 1

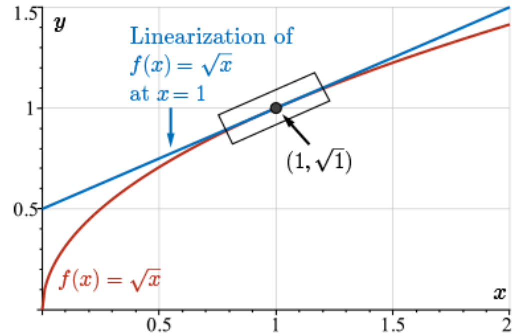
Use the linearization of $f(x) = \sqrt{x}$ at $x = 1$ to approximate the value of $\sqrt{1.1}$.

Solution

We know that if we zoom in enough on the graph of a smooth curve, then the curve is close to a straight line.

As such, the linearization of $f(x)$ at $x = 1$ should provide a reasonably good approximation for the values of $f(x)$ for $x \approx 1$.

Since $f(x) = \sqrt{x}$, we have $f'(x) = \dfrac{1}{2\sqrt{x}}$, so $f(1) = \sqrt{1} = 1$ and $f'(1) = \dfrac{1}{2\sqrt{1}} = \dfrac{1}{2}$.



Therefore, the linearization of f at $x = 1$ is

$$L(x) = f(1) + f'(1)(x-1) = 1 + \frac{1}{2}(x-1)$$

Using Linearization to Approximate Square Roots

Example 1

Use the linearization of $f(x) = \sqrt{x}$ at $x = 1$ to approximate the value of $\sqrt{1.1}$.

Solution

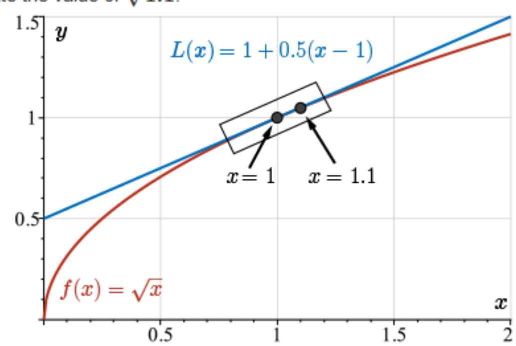
As we can see in the picture, when $x = 1.1$, we have $L(1.1) \approx f(1.1)$.
We can easily calculate

$$L(1.1) = 1 + \frac{1}{2}(1.1-1) = 1.05$$

So, we estimate that

$$\sqrt{1.1} = f(1.1) \approx L(1.1) = 1.05$$

Checking with a calculator, we get $\sqrt{1.1} \approx 1.0488\ldots$ and so the approximation is correct up to two decimal places.



Using Linearization to Approximate Square Roots

Example 2

Use the linearization of $f(x) = \sqrt{x}$ at $x = 1$ to approximate the value of $\sqrt{2}$.

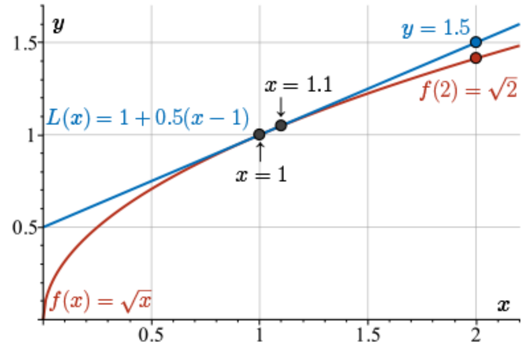
Solution

From the previous example, the linearization of $f(x) = \sqrt{x}$ at $x = 1$ is given by

$$L(x) = 1 + \frac{1}{2}(x - 1)$$

Since $x = 2$ is farther from $x = 1$, we would expect this linearization to give a worse approximation of $\sqrt{2}$ than it gave for $\sqrt{1.1}$.

This is apparent in the graph shown.



Here, we have

$$L(2) = 1 + \frac{1}{2}(2 - 1) = 1 + \frac{1}{2} = 1.5$$

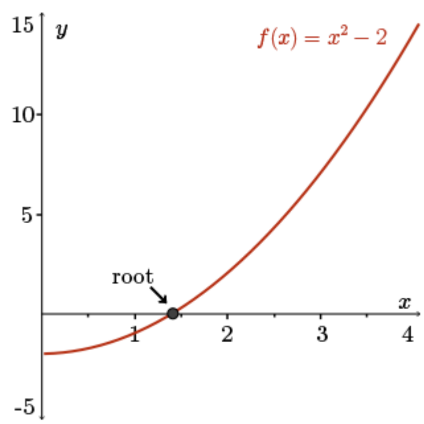
Using a calculator, $\sqrt{2} \approx 1.414$, so the linear approximation $\sqrt{2} = f(2) \approx L(2)$ is not correct up to a single decimal place. (Recall that the approximation of $\sqrt{1.1} = f(1.1) \approx L(1.1)$ was correct up to two decimal places.)

How do we approximate $\sqrt{2}$ more accurately?

Newton's Method

Newton's method can be used to approximate the roots of equations of the form $f(x) = 0$.

In particular, it can be used to approximate the positive root of the equation $x^2 - 2 = 0$ to any degree of accuracy we would like. (This is, of course, equivalent to approximating the value of $\sqrt{2}$ to any degree of accuracy.)



Newton's Method

We will now introduce the theory behind Newton's method, while attempting to approximate the positive root of the equation $x^2 - 2 = 0$ more accurately.

First, we guess that the positive root of the equation $f(x) = x^2 - 2 = 0$ is near $x = 3$.

Let's find the linearization of $f(x)$ at $x = 3$:

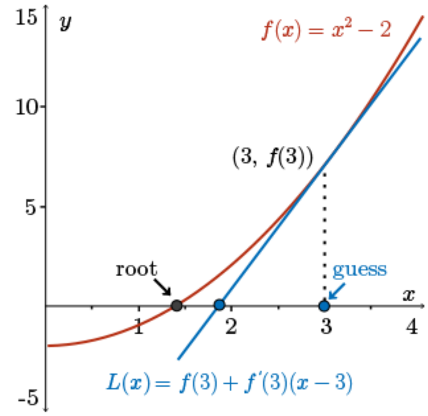
$$L(x) = f(3) + f'(3)(x-3) = 7 + 6(x-3)$$

Then, we have $f(x) \approx L(x)$ for $x \approx 3$.

Since $L(x)$ is our approximation of $f(x)$, we use the x-intercept of $L(x)$ to approximate the x-intercept of $f(x)$ (that is, the root of $f(x) = 0$).

If the root of $f(x) = 0$ is indeed near $x = 3$, then this approximation will be useful.

Here, we see a graphical representation of this process.



Setting $L(x) = 0$ and solving for the x-intercept, we get

$$0 = 7 + 6(x-3) \implies x - 3 = -\frac{7}{6} \implies x = 3 - \frac{7}{6} = \frac{11}{6}$$

We take $x = \frac{11}{6}$ as our "updated approximation" of the root.

Now, we repeat this process, this time with $x = \frac{11}{6}$ instead of $x = 3$.

Newton's Method

To keep track of the process, we will introduce some notation:

Let $x_1 = 3$ denote our first guess, and let $x_2 = \dfrac{11}{6}$ denote our second approximation.

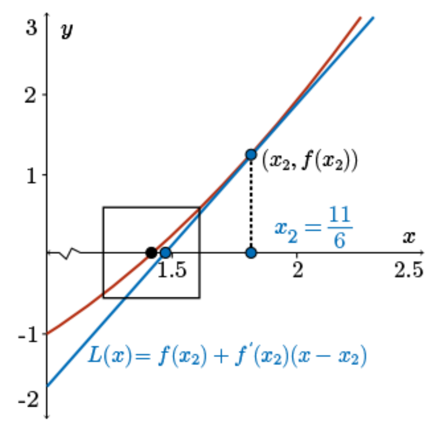
Now, we repeat the same process with x_2.

First, we find the linearization of f at $x_2 = \dfrac{11}{6}$.

We have $f\left(\dfrac{11}{6}\right) = \left(\dfrac{11}{6}\right)^2 - 2 = \dfrac{121}{36} - 2 = \dfrac{49}{36}$ and $f'\left(\dfrac{11}{6}\right) = 2\left(\dfrac{11}{6}\right) = \dfrac{11}{3}$

and so the linearization is

$$L(x) = f(x_2) + f'(x_2)(x - x_2) = \frac{49}{36} + \frac{11}{3}\left(x - \frac{11}{6}\right)$$



Newton's Method

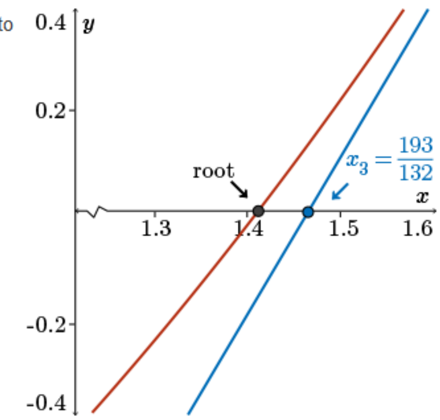
Again, as we see in the picture, the x-intercept of this line is "closer" to the desired root than our second approximation $x_2 = \frac{11}{6}$.

By setting $y = 0$ and solving for x, we get

$$0 = \frac{49}{36} + \frac{11}{3}\left(x - \frac{11}{6}\right)$$
$$\implies x - \frac{11}{6} = -\left(\frac{49}{36}\right)\left(\frac{3}{11}\right)$$
$$\implies x = \frac{11}{6} - \frac{49}{132} = \frac{193}{132}$$

and hence the x-intercept is $\frac{193}{132} \approx 1.46$.

This will be our next approximation $x_3 = \frac{193}{132}$.



Newton's Method

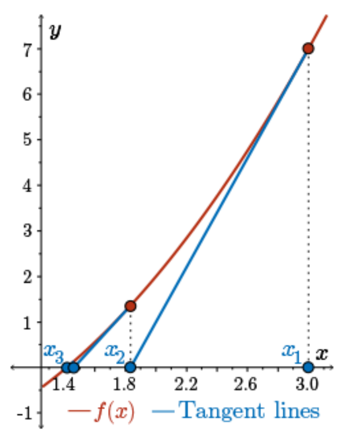
If we repeat this process, then we get the following sequence of numbers:

$$x_1 = 3$$
$$x_2 = \frac{11}{6} \approx 1.833$$
$$x_3 = \frac{193}{132} \approx 1.462$$
$$x_4 = \frac{72097}{50952} \approx 1.415$$
$$x_5 = \frac{10390190017}{7346972688} \approx 1.414$$

Notice that the 5^{th} approximation, x_5, is correct up to at least three decimal places.

In fact, if you check on your calculator, x_5 agrees with the value of $\sqrt{2}$ up to six decimal places!

$$x_5 = \frac{10390190017}{7346972688} = 1.41421378\ldots$$
$$\sqrt{2} = 1.41421356\ldots$$



This method for approximating roots of equations is called **Newton's method** (or the **Newton-Raphson method**).

Newton's Method

Let's recap this process for finding roots of an equation $f(x) = 0$:

1. Make a guess at a root of f and call this guess x_1.
2. Find the tangent line approximation to f at x_1:

$$y = f(x_1) + f'(x_1)(x - x_1)$$

and find the x-intercept of this line:

$$\begin{aligned} 0 &= f(x_1) + f'(x_1)(x - x_1) \\ \implies x - x_1 &= -\frac{f(x_1)}{f'(x_1)} \\ \implies x &= x_1 - \frac{f(x_1)}{f'(x_1)} \end{aligned}$$

3. Let $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$, the x-intercept found in step 2), and return to step 1) with x_2 in place of x_1.

Newton's Method

Repeating this process, we generate the following sequence of approximations:

$$\begin{aligned} x_1 &= \text{guess at root} \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\ x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} \\ &\vdots \end{aligned}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Under which conditions does Newton's method produce a sequence that converges to a root of $f(x) = 0$?

Newton's Method

Example 3

Starting with $x_1 = 1$, find the 3^{rd} approximation x_3 to the positive root of the equation $x^2 + x - 1 = 0$. Check the accuracy of the approximation using the quadratic formula.

Solution

Let $f(x) = x^2 + x - 1$, then $f'(x) = 2x + 1$.

$$x_1 = 1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{3} = \frac{2}{3}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{2}{3} - \frac{f\left(\frac{2}{3}\right)}{f'\left(\frac{2}{3}\right)} = \frac{2}{3} - \frac{\frac{1}{9}}{\frac{7}{3}} = \frac{13}{21}$$

Using the quadratic formula, we get

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$$

So, the positive root is

$$x = \frac{-1 + \sqrt{5}}{2} \approx 0.618$$

Newton's Method

Example 3

Starting with $x_1 = 1$, find the 3^{rd} approximation x_3 to the positive root of the equation $x^2 + x - 1 = 0$. Check the accuracy of the approximation using the quadratic formula.

Solution

$$x = \frac{-1 + \sqrt{5}}{2} \approx 0.618$$

Since $x_3 = \dfrac{13}{21} \approx 0.619$, our approximation was correct up to two decimal places.

What happens if we start with $x_1 = -1$ instead? What about $x_1 = 0$?

Remarks

For certain functions $f(x)$ and guesses x_1, the sequence of numbers $x_1, x_2, x_3, \ldots$ will converge to a root of the equation $f(x) = 0$ quite quickly (like in the previous examples).

However, in many cases, the sequence may converge very slowly, or not at all.
This is often the case if the tangent line to f at x_1 is close to horizontal.
Can you explain why?
What happens if the equation, $f(x) = 0$, has more than one root?
Will different guesses converge to different roots?

Newton's Method

Example 4

The equation $x^3 - 4x - 1 = 0$ has three real roots, exactly one of which is positive. Find the value of the positive root correct to five decimal places.

Solution

First, we check if the equation has any rational roots.

By the rational roots theorem, the only possible rational roots are ± 1 and so we check $f(-1) = (-1)^3 - 4(-1) - 1 = 2$ and $f(1) = (1)^3 - 4(1) - 1 = -4$, and conclude that the equation has no rational roots.

Since we cannot factor the cubic polynomial easily, we will use Newton's method to locate the positive root.
Let $f(x) = x^3 - 4x - 1$, then $f'(x) = 3x^2 - 4$.

Therefore, the formula for Newton's method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 4x_n - 1}{3x_n^2 - 4}$$

for all $n \geq 1$.

Since we are looking for a positive root, let's start with $x_1 = 1$.

Newton's Method

Example 4

The equation $x^3 - 4x - 1 = 0$ has three real roots, exactly one of which is positive. Find the value of the positive root correct to five decimal places.

Solution

We generate the following sequence of approximations:

$$x_2 = x_1 - \frac{x_1^3 - 4x_1 - 1}{3x_1^2 - 4} = 1 - \frac{(1)^3 - 4(1) - 1}{3(1)^2 - 4} = 1 - \frac{(-4)}{(-1)} = 1 - 4 = -3$$

$$x_3 = x_2 - \frac{x_2^3 - 4x_2 - 1}{3x_2^2 - 4} = -3 - \frac{(-3)^3 - 4(-3) - 1}{3(-3)^2 - 4} = -3 - \frac{(-16)}{(23)} = -3 + \frac{16}{23} = -\frac{53}{23} \approx -2.30435$$

$x_4 \approx -1.96749$

$x_5 \approx -1.86947$

$x_6 \approx -1.86087$

The sequence seems to be approaching a number around -1.86, and so we suspect that we have chosen a bad guess x_1 for finding the positive root.

In fact, this sequence is approaching one of the negative roots of the equation.

To determine a better initial guess, we need further analysis to narrow down the location of the positive root.

Newton's Method

Example 4

The equation $x^3 - 4x - 1 = 0$ has three real roots, exactly one of which is positive. Find the value of the positive root correct to five decimal places.

Solution

If we evaluate the function, f, at various small integer values, $x = 0, 1, 2, 3, \ldots$, we get the following:

$$\begin{aligned} f(0) &= -1 \\ f(1) &= -4 \\ f(2) &= -1 \\ f(3) &= 14 \end{aligned}$$

Since $f(2)$ is negative and $f(3)$ is positive, we conclude that we must have a root in the interval $[2, 3]$.

The continuous polynomial must cross the x-axis at some point between $x = 2$ and $x = 3$.
Therefore, there must be some number a in the interval $(2, 3)$ such that $f(a) = 0$.
A formal justification of this fact is the content of what is called the intermediate value theorem.
This theorem is generally presented in any university calculus class.

Since $f(2) = -1$ is much closer to zero than $f(3) = 14$, we suspect that the root may be closer to 2 and so we choose $x_1 = 2$.

Newton's Method

Example 4

The equation $x^3 - 4x - 1 = 0$ has three real roots, exactly one of which is positive. Find the value of the positive root correct to five decimal places.

Solution

Again, we apply Newton's method.

$$x_1 = 2$$

$$x_2 = x_1 - \frac{x_1^3 - 4x_1 - 1}{3x_1^2 - 4} = 2 - \frac{(2)^3 - 4(2) - 1}{3(2)^2 - 4} = 2 - \frac{(-1)}{(8)} = 1 + \frac{1}{8} = \frac{17}{8} = 2.125$$

$$x_3 \approx 2.114975450$$

$$x_4 \approx 2.114907545$$

$$x_5 \approx 2.114907541$$

Since x_4 and x_5 agree on (at least) five decimal places, we conclude that the positive root of the equation, to five decimal places of accuracy, is $x = 2.11491$.

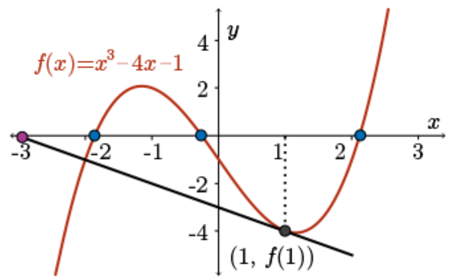
Newton's Method

Example 4

The equation $x^3 - 4x - 1 = 0$ has three real roots, exactly one of which is positive. Find the value of the positive root correct to five decimal places.

Solution

Why was $x_1 = 1$ a bad guess, while $x_1 = 2$ was a good guess?
Generally, the best guesses are numbers x_1 that are relatively close to the root in question.
What does relatively close mean? That will depend on the function being considered.
Consider the graph of the cubic $f(x) = x^3 - 4x - 1$.
The three roots lie in the intervals $[-2, -1]$, $[-1, 0]$, and $[2, 3]$.
(Note that we can also locate the intervals in which the two negative roots lie by evaluating the function f at various negative points $x = -1, -2, -3, \ldots$)



If we choose $x_1 = -2$ or $x_1 = -1$, then the sequence approaches the root in the interval $[-2, -1]$;
if we choose $x_1 = 0$, then the sequence approaches the root in the interval $[-1, 0]$;
if we choose $x_1 = 2$ or $x_1 = 3$, then the sequence approaches the positive root.

Can you see why $x_1 = 1$ generated a sequence approaching the root in the interval $[-2, -1]$?

Newton's Method

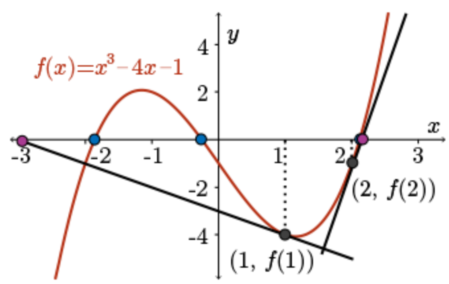
Example 4

The equation $x^3 - 4x - 1 = 0$ has three real roots, exactly one of which is positive. Find the value of the positive root correct to five decimal places.

Solution

x_1	Correct to two decimal places
-1000	$x_{19} = -1.86\ldots$
-100	$x_{14} = -1.86\ldots$
-30	$x_{11} = -1.86\ldots$
-5	$x_7 = -1.86\ldots$
-3	$x_5 = -1.86\ldots$
-2	$x_3 = -1.86\ldots$
-1.2	$x_8 = -1.86\ldots$
-1.1	$x_7 = 2.11\ldots$
-1	$x_7 = -1.86\ldots$
-0.9	$x_4 = -0.25\ldots$
-0.5	$x_3 = -0.25\ldots$
0	$x_2 = -0.25\ldots$

x_1	Correct to two decimal places
0.5	$x_4 = -0.25\ldots$
0.8	$x_7 = -0.25\ldots$
0.9	$x_5 = -1.86\ldots$
1	$x_6 = -1.86\ldots$
1.1	$x_9 = -1.86\ldots$
1.2	$x_{10} = 2.11\ldots$
2	$x_3 = 2.11\ldots$
3	$x_5 = 2.11\ldots$
5	$x_6 = 2.11\ldots$
30	$x_{11} = 2.11\ldots$
100	$x_{14} = 2.11\ldots$
1000	$x_{20} = 2.11\ldots$



Newton's Method

Example 4

The equation $x^3 - 4x - 1 = 0$ has three real roots, exactly one of which is positive. Find the value of the positive root correct to five decimal places.

Solution

Observe that the success of Newton's method can be very sensitive to small changes in the initial guess, x_1.

(For example, $x_1 = 1.2$ would have been successful in locating the positive root, but $x_1 = 1.1$ would have failed.)

In fact, there are cases where a small change in the guess x_1 can mean the difference between a sequence which approaches a root, and a sequence that does not approach any number at all.

You will explore such examples in the student exercises.