Applications Of The Dot And Cross products

In This Module

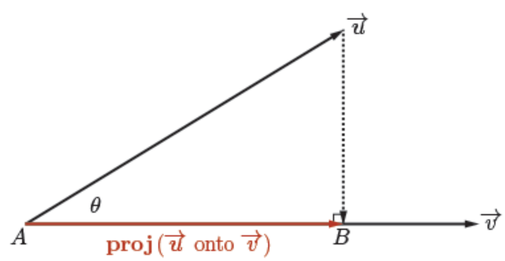
While the dot product and cross product may seem to be simply abstract mathematical concepts, they have a wide range of interesting geometrical applications, which have been very useful in fields such as physics.

Projections

Given two vectors, $\vec{u}$ and $\vec{v}$, placed tail to tail with angle θ between them, drop a perpendicular from the tip of $\vec{u}$ to the line containing $\vec{v}$.

The vector lying along the line containing $\vec{v}$, which has magnitude equal to the component of $\vec{u}$ in the direction of $\vec{v}$ (i.e., AB in our diagram), is called the **vector projection** of $\vec{u}$ onto $\vec{v}$.

To find the magnitude of this new vector, $\mathbf{proj}\,(\vec{u} \text{ onto } \vec{v})$, we can use simple trigonometric ratios.



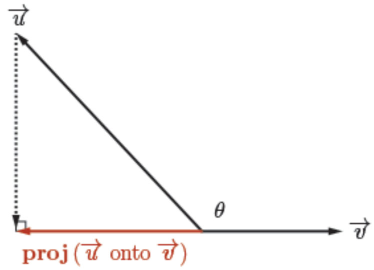
$$
\begin{aligned}
|\cos(\theta)| &= \frac{|\mathbf{proj}\,(\vec{u} \text{ onto } \vec{v})|}{|\vec{u}|} \\
|\mathbf{proj}\,(\vec{u} \text{ onto } \vec{v})| &= |\vec{u}||\cos(\theta)| \\
&= \frac{|\vec{u}||\vec{v}||\cos(\theta)|}{|\vec{v}|} \\
&= \frac{||\vec{u}||\vec{v}|\cos(\theta)|}{|\vec{v}|} \\
&= \frac{|\vec{u}\cdot\vec{v}|}{|\vec{v}|}
\end{aligned}
$$

Projections

Note: If $90^\circ < \theta < 180^\circ$, then $\cos(\theta) < 0$.

In this case, the projection vector would have direction opposite to $\vec{v}$.

We need magnitude (absolute value bars) around $\cos(\theta)$ in the derivation to ensure that the **magnitude** of the vector is positive when $90^\circ < \theta < 180^\circ$.

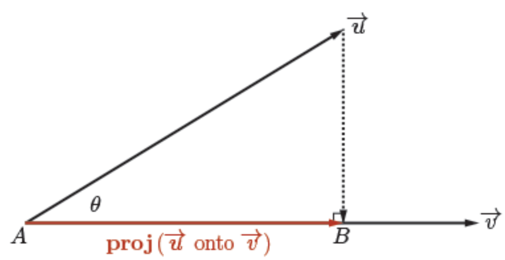


Projections

$$|\mathbf{proj}\ (\vec{u} \text{ onto } \vec{v})| = \frac{|\vec{u} \cdot \vec{v}|}{|\vec{v}|}$$

Thus the vector projection of $\vec{u}$ onto $\vec{v}$ is a vector with magnitude $\frac{|\vec{u} \cdot \vec{v}|}{|\vec{v}|}$ and direction along the line containing $\vec{v}$.

$$\begin{aligned} \therefore \mathbf{proj}\ (\vec{u} \text{ onto } \vec{v}) &= \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}\right) \hat{v} \\ &= \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}\right) \frac{\vec{v}}{|\vec{v}|} \\ &= \frac{(\vec{u} \cdot \vec{v})\vec{v}}{|\vec{v}|^2} \end{aligned}$$



where we drop the magnitude bars on $\vec{u} \cdot \vec{v}$ to allow for the possibility that the projection vector is in a direction opposite to $\vec{v}$.

At this point, it is worth defining the **scalar projection** of $\vec{u}$ onto $\vec{v}$ as the signed (positive or negative) magnitude of the vector projection of $\vec{u}$ onto $\vec{v}$.

Then as we previously observed, the scalar projection of $\vec{u}$ onto $\vec{v}$ is equal to $|\vec{u}| \cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$.

Summary

The applications of vector projections are many and varied. They exist in areas such as engineering, quantum mechanics, digital video and audio recording, computer animation, and statistics to name only a few.

The **vector projection** of $\vec{u}$ onto $\vec{v}$ is

$$\textbf{proj}\,(\vec{u} \text{ onto } \vec{v}) = \frac{(\vec{u} \cdot \vec{v})\vec{v}}{|\vec{v}|^2}$$

The **magnitude of the vector projection** of $\vec{u}$ onto $\vec{v}$ is

$$\frac{|\vec{u} \cdot \vec{v}|}{|\vec{v}|}$$

Examples

Example 1

Calculate the projection of $\vec{u} = (1, -2, 3)$ onto $\vec{v} = (3, 0, 4)$ and determine its magnitude.

Solution

Vector:

$$\begin{aligned}
\textbf{proj}\,(\vec{u} \text{ onto } \vec{v}) &= \frac{(\vec{u} \cdot \vec{v})\vec{v}}{|\vec{v}|^2} \\
&= \frac{(15)(3, 0, 4)}{25} \\
&= \left(\frac{9}{5}, 0, \frac{12}{5}\right)
\end{aligned}$$

Magnitude:

$$|\textbf{proj}\,(\vec{u} \text{ onto } \vec{v})| = \frac{|\vec{u} \cdot \vec{v}|}{|\vec{v}|} = \frac{15}{5} = 3$$

Or alternatively,

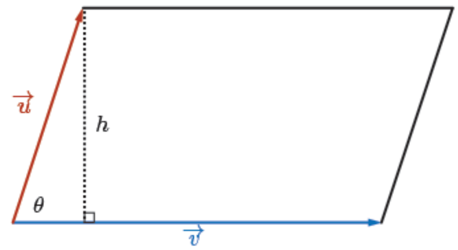
$$|\textbf{proj}\,(\vec{u} \text{ onto } \vec{v})| = \left|\left(\frac{9}{5}, 0, \frac{12}{5}\right)\right| = \sqrt{\left(\frac{9}{5}\right)^2 + (0)^2 + \left(\frac{12}{5}\right)^2} = \sqrt{\frac{225}{25}} = \sqrt{9} = 3$$

Area of a Parallelogram

The magnitude of the cross product possesses a surprising connection with geometry. It is equal to the area of the completed parallelogram formed by two vectors, $\vec{u}$ and $\vec{v}$.

Recall:

Area of a parallelogram = base × perpendicular height



From the diagram, the length of the base is $|\vec{v}|$ and since $\sin(\theta) = \dfrac{h}{|\vec{u}|}$, then $h = |\vec{u}|\sin(\theta)$.

Substituting these gives $A = |\vec{v}||\vec{u}|\sin(\theta) = |\vec{u} \times \vec{v}|$.

The area of a parallelogram is

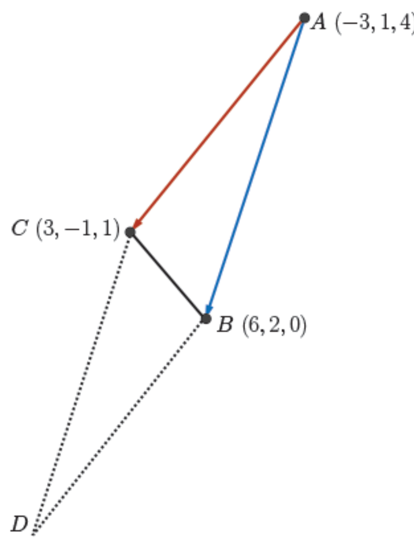
$$A = |\vec{u} \times \vec{v}|$$

where $\vec{u}$ and $\vec{v}$ are adjacent sides of a parallelogram.

Examples

Example 2

Find the area of a triangle with vertices $A\,(-3, 1, 4)$, $B\,(6, 2, 0)$, and $C\,(3, -1, 1)$.



Solution

Draw the diagram and label the vertices.

Next, we determine the vectors between the vertices, and then apply the formula for the area of the parallelogram.

Since a diagonal of a parallelogram bisects its area, the area of $\triangle ABC$ is one half the area of parallelogram $ABDC$.

$\overrightarrow{AC} = (6, -2, -3)$ and $\overrightarrow{AB} = (9, 1, -4)$.

Determining the cross product:

$$\overrightarrow{AC} \times \overrightarrow{AB} = (11, -3, 24)$$

Quickly check that

$$(6, -2, -3) \cdot (11, -3, 24) = 66 + 6 - 72 = 0$$
$$(9, 1, -4) \cdot (11, -3, 24) = 99 - 3 - 96 = 0$$

Thus, $\text{Area}_{\triangle ABC} = \frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{AB}\right| = \frac{1}{2}\sqrt{(11)^2 + (-3)^2 + (24)^2} = \frac{1}{2}\sqrt{121 + 9 + 576} = \frac{1}{2}\sqrt{706}$.

Triple Scalar Product

Another interesting connection between algebraic operations on vectors and geometry is the **triple scalar product** of three vectors, $\vec{a}, \vec{b}$, and $\vec{c}$, which is defined as

$$\vec{c} \cdot \left(\vec{a} \times \vec{b}\right)$$

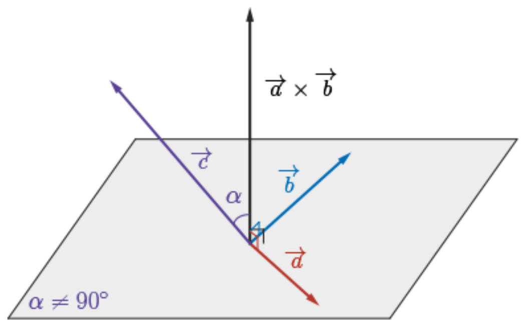
Note that this is a scalar quantity.

In addition to other applications, the triple scalar product is often used to determine if three vectors are coplanar (lie in the same plane).

How so?
Consider that any two vectors, $\vec{a}$ and $\vec{b}$, are coplanar.
Assume that these two vectors, $\vec{a}$ and $\vec{b}$, are not collinear.
Therefore, their cross product, $\vec{a} \times \vec{b}$, is orthogonal to the plane in which $\vec{a}$ and $\vec{b}$ lie.

So, if $\vec{c} \cdot \left(\vec{a} \times \vec{b}\right) = 0$, then $\vec{c}$ is orthogonal to $\vec{a} \times \vec{b}$, and so $\vec{c}$ must lie in the same plane as $\vec{a}$ and $\vec{b}$.

Similarly, if $\vec{c} \cdot \left(\vec{a} \times \vec{b}\right) \neq 0$, then $\vec{c}$ is not orthogonal to $\vec{a} \times \vec{b}$, and so $\vec{a}, \vec{b}$, and $\vec{c}$ are not coplanar.

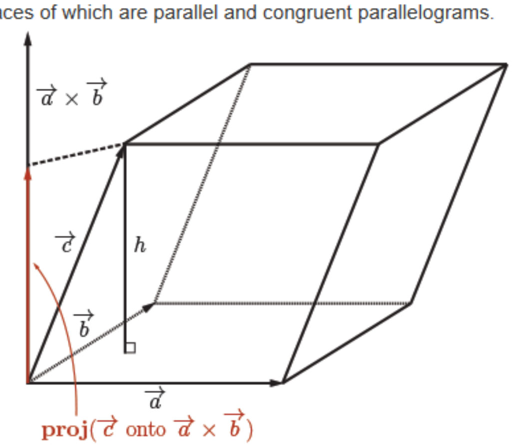


Volume of a Parallelepiped

A **parallelepiped** is a box-like solid, where the opposite faces of which are parallel and congruent parallelograms.
Let $\vec{a}, \vec{b}$, and $\vec{c}$ be three vectors whose tails meet at one vertex of the parallelepiped.
The absolute value of the triple scalar product of these three vectors gives the volume of the parallelepiped.

From the diagram, we see

$$\begin{aligned}
\text{Volume} &= (\text{Area of base}) \times h \\
\text{Area of base} &= \left|\vec{a} \times \vec{b}\right| \\
h &= \left|\mathbf{proj}\ (\vec{c} \text{ onto } (\vec{a} \times \vec{b}))\right| \\
&= \frac{\left|\vec{c} \cdot (\vec{a} \times \vec{b})\right|}{\left|\vec{a} \times \vec{b}\right|} \\
\therefore\ \text{Volume} &= \left|\vec{a} \times \vec{b}\right| \left(\frac{\left|\vec{c} \cdot (\vec{a} \times \vec{b})\right|}{\left|\vec{a} \times \vec{b}\right|}\right) \\
&= \left|\vec{c} \cdot (\vec{a} \times \vec{b})\right|
\end{aligned}$$



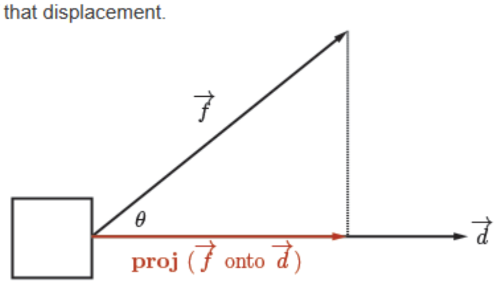
Question: Is $\left|\vec{c} \cdot (\vec{a} \times \vec{b})\right|$ equivalent to $\left|\vec{a} \cdot (\vec{b} \times \vec{c})\right|$?

Work

Work is said to have been done when a force acts on an object to displace it in the direction of the force.

Mathematically, work is defined to be the scalar quantity equal to the component of the force that acts in the direction of the displacement multiplied by the magnitude of that displacement.

- $\vec{f}$ is the force acting on the object.
- $\vec{d}$ is the displacement of the object.
- θ is the angle between $\vec{f}$ and $\vec{d}$.



$$\begin{aligned}\text{Work} &= \left|\textbf{proj}\,(\vec{f}\text{ onto }\vec{d})\right|\,|\vec{d}| \\ &= |\vec{f}\,|\cos(\theta)|\vec{d}|, \qquad \text{for } 0^\circ \le \theta \le 90^\circ \\ &= |\vec{f}\,||\vec{d}|\cos(\theta) \\ &= \vec{f}\cdot\vec{d}\end{aligned}$$

Note that the final definition of work is the dot product, $\vec{f}\cdot\vec{d}$, of the force and displacement vectors, and **not** the magnitude.

This allows for the possibility that work may be a negative quantity. Note that this happens when $\vec{d}$ and the $\textbf{proj}\left(\vec{f}\text{ onto }\vec{d}\right)$ are in opposite directions.

The units of measurement for work are defined to be **newton-metres (Nm)** or **joules (J)**, where 1 Nm $= 1$ J, so long as displacement is measured in metres and force in newtons.

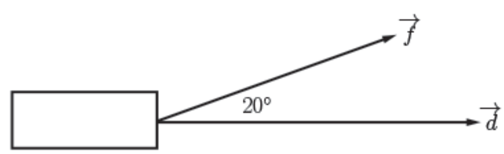
Examples

Example 3

A wagon is pulled a distance of 200 m by a 160 N force applied at an angle of 20° to the road. Calculate the work done by the force.

Solution

$$\begin{aligned}W &= |\vec{f}\,||\vec{d}|\cos(\theta) \\ &= (160)(200)\cos(20^\circ) \\ &\approx 30\,070\text{ J}\end{aligned}$$



Examples

Example 4

Find the amount of work done by a 5 N force in moving an object from $A\,(-2,1)$ to $B\,(7,8)$, where the force is applied at a $30°$ angle to AB at A. Assume the distance moved is in metres.

Solution

First, we determine the direction vector

$$\overrightarrow{AB} = (9,7)$$

Therefore,

$$\begin{aligned} W &= |\vec{f}||\vec{d}|\cos(\theta) \\ &= (5)\left(\sqrt{9^2+7^2}\right)\cos(30°) \\ &= (5)(\sqrt{130})\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{5}{2}\sqrt{390}\ \mathrm{J} \end{aligned}$$