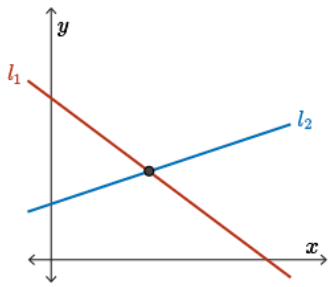
# The Intersection Of Two Lines In R2 And R3

## Intersection of Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$

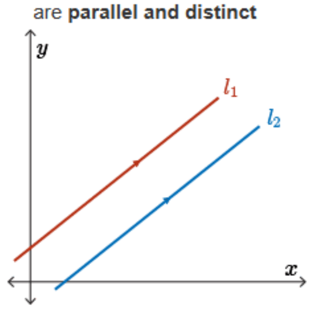
### 2 Dimensions

There are three cases to consider with respect to the way two lines may intersect in $\mathbb{R}^2$ .

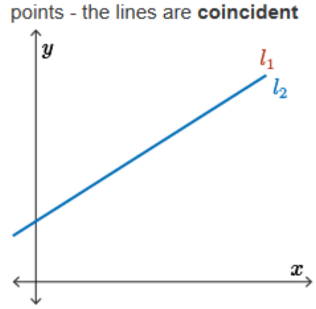
1. Intersect in exactly one point




2. No intersection points - the lines are **parallel and distinct**




3. Infinite number of intersection points - the lines are **coincident**




## Intersection of Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$

### 3 Dimensions

The three cases in which two lines may intersect in $\mathbb{R}^2$ also exist in $\mathbb{R}^3$. The two lines may

- intersect in exactly one point,
- be parallel and distinct and not intersect, or
- be coincident and intersect in an infinite number of points.

However, there is one additional possibility in $\mathbb{R}^3$ not found in $\mathbb{R}^2$: skew lines.

Two lines in $\mathbb{R}^3$ are said to be **skew lines** if they are not parallel and do not intersect. Equivalently, they are lines that are not coplanar.

## Examples

### Example 1

Find the points of intersection of the following lines.

**a.** In $\mathbb{R}^2$: $l_1: 2x - 3y + 8 = 0$

$l_2: 2x - 3y - 1 = 0$

**Solution**

**Method 1**

Since the normal vectors for the two lines are equal ($\overrightarrow{n_1} = (2, -3) = \overrightarrow{n_2}$) and the constant terms are not equal ($8 \neq -1$), then $l_1$ and $l_2$ are parallel and distinct.

Thus, the two lines, $l_1$ and $l_2$, do not intersect.

**Method 2**

Using elimination, we subtract the two equations to get $9 = 0$.

This statement is clearly not true and is independent of the values of $x$ and $y$.

Hence, the two lines share no common point and thus do not intersect.

Since these are lines in $\mathbb{R}^2$, they must be parallel and distinct.

If a linear system of equations has no solutions, such as the system in part **a**, then the system is said to be **inconsistent**.

If a linear system has at least $1$ solution, then it is said to be **consistent**.

## Examples

### Example 1

Find the points of intersection of the following lines.

**b.** In $\mathbb{R}^2$: $l_1: x - 5y + 6 = 0$

$l_2: 3x + 10y - 7 = 0$

**Solution**

Recognizing that the normal vectors are not scalar multiples of one another ($l_1$ and $l_2$ are not parallel), we again use the method of elimination.

Multiplying the first equation by $2$ and adding the two equations, we get

$$\begin{array}{rrrrrrrr} & 2x & - & 10y & + & 12 & = & 0 \\ + & 3x & + & 10y & - & 7 & = & 0 \\ \hline & 5x & & & + & 5 & = & 0 \end{array}$$

so $x = -1$.

Substituting $x = -1$ into the first equation, we solve to get $y = 1$.

Hence, the two lines intersect at the point $(-1, 1)$.

This is an example of a linear system of equations that is consistent.

## Examples

### Example 1

Find the points of intersection of the following lines.

**c.** In $\mathbb{R}^3$: $l_1$: $(x, y, z) = (9, 3, 4) + t\,(4, 0, 2),\ t \in \mathbb{R}$

$l_2$: $x = 3r - 2,\ \ y = 6 - 3r,\ \ z = r - 1,\ r \in \mathbb{R}$

**Solution**

There are many different approaches for solving systems of this form.

**Method 1**

In this first method, we will solve by converting both lines into parametric equations and determining the values of the parameters $t$ and $r$.

Converting $l_1$ into parametric form gives $x = 9 + 4t,\ \ y = 3$, and $z = 4 + 2t$.

Equating the parametric equations for $x$, we get

$$\begin{aligned} 9 + 4t &= 3r - 2 \\ \therefore 3r &= 4t + 11 \qquad (1) \end{aligned}$$

Equating the parametric equations for $y$, we get

$$\begin{aligned} 3 &= 6 - 3r \\ 3r &= 3 \\ \therefore r &= 1 \end{aligned}$$

Substituting $r = 1$ into the equation for $l_2$ gives $x = 3(1) - 2 = 1, y = 6 - 3(1) = 3$, and $z = 1 - 1 = 0$.


## Examples

### Example 1

Find the points of intersection of the following lines.

**c.** In $\mathbb{R}^3$: $l_1$: $(x, y, z) = (9, 3, 4) + t\,(4, 0, 2),\ t \in \mathbb{R}$

$l_2$: $x = 3r - 2,\ \ y = 6 - 3r,\ \ z = r - 1,\ r \in \mathbb{R}$

**Solution**

There are many different approaches for solving systems of this form.


We must verify that this point $(1, 3, 0)$ also lies on $l_1$.

By substituting $r = 1$ into equation $(1)$, we can find the value of the parameter $t$ for $l_1$.

$$\begin{aligned} 3r &= 4t + 11 \qquad (1) \\ 3(1) &= 4t + 11 \\ 4t &= -8 \\ t &= -2 \end{aligned}$$

Substituting $t = -2$ into the equation for $l_1$ gives $(x, y, z) = (1, 3, 0)$.

Therefore, $l_1$ and $l_2$ intersect at the point $(1, 3, 0)$.

## Examples

### Example 1

Find the points of intersection of the following lines.

**c.** In $\mathbb{R}^3$: $l_1$: $(x, y, z) = (9, 3, 4) + t\,(4, 0, 2),\ t \in \mathbb{R}$

$l_2$: $x = 3r - 2,\ \ y = 6 - 3r,\ \ z = r - 1,\ r \in \mathbb{R}$

**Solution**

**Method 2**

Another way to solve this system is to write the equation of $l_1$ in symmetric form, and then substitute the parametric equations of $l_2$ into $l_1$.

To this effect, the symmetric equations of $l_1$ are

$$l_1: \quad \frac{x-9}{4} = \frac{z-4}{2},\ \ y = 3$$

Substituting $l_2$ ($x = 3r - 2$ and $z = r - 1$) into $l_1$ and solving for $r$, we get

$$\begin{aligned} \frac{(3r-2)-9}{4} &= \frac{(r-1)-4}{2} \\ \frac{3r-11}{4} &= \frac{r-5}{2} \\ 6r - 22 &= 4r - 20 \\ 2r &= 2 \\ r &= 1 \end{aligned}$$

## Examples

### Example 1

Find the points of intersection of the following lines.

**c.** In $\mathbb{R}^3$: $l_1$: $(x, y, z) = (9, 3, 4) + t\,(4, 0, 2),\ t \in \mathbb{R}$

$l_2$: $x = 3r - 2,\ \ y = 6 - 3r,\ \ z = r - 1,\ r \in \mathbb{R}$

**Solution**

**Method 2**

$$l_1: \quad \frac{x-9}{4} = \frac{z-4}{2},\ \ y = 3$$

When $r = 1$, the $x$ and $z$ coordinates of $l_2$ are $x = 3(1) - 2 = 1$ and $z = 1 - 1 = 0$.

These $(x = 1, z = 0)$ also satisfy the symmetric equations of $l_1$, since $\dfrac{1-9}{4} = \dfrac{0-4}{2}$.

We must still verify that the $y$ coordinate of $l_1$, $y = 3$, also satisfies $l_2$ when $r = 1$.

Substituting, we get $y = 6 - 3r = 6 - 3(1) = 3$.

Therefore, the point of intersection is $(1, 3, 0)$.

## Examples

### Example 1

Find the points of intersection of the following lines.

**d.** In $\mathbb{R}^3$: $l_1 : \vec{r} = (3,1,-1) + t\,(2,1,-3),\ t \in \mathbb{R}$ and $l_2 : \dfrac{x-7}{2} = \dfrac{y-12}{-4} = \dfrac{z-17}{5}$

**Solution**

**Method 1**

The parametric equations for $l_1$ are $x = 3 + 2t,\ y = 1 + t,\ z = -1 - 3t,\ t \in \mathbb{R}$.
To convert the symmetric equations for $l_2$ to parametric equations, equate each expression with $r$ (since they are equal to one another, they are equal to some parameter $r \in \mathbb{R}$) and solve each for the appropriate coordinate; that is

$$r = \frac{x-7}{2} \qquad r = \frac{y-12}{-4} \qquad r = \frac{z-17}{5}$$

$$x = 2r + 7 \qquad y = -4r + 12 \qquad z = 5r + 17$$

Equating the corresponding parametric equations for $l_1$ and $l_2$, we get

$$3 + 2t = 2r + 7 \qquad 1 + t = -4r + 12 \qquad -1 - 3t = 5r + 17$$

$$t - r = 2 \quad (1) \qquad t + 4r = 11 \quad (2) \qquad 3t + 5r = -18 \quad (3)$$

We have a system of three linear equations in two unknowns, $t$ and $r$.
We may, again, use the method of elimination to solve for $t$ and $r$.
Subtracting equations $(1)$ and $(2)$, we get $-5r = -9$ and so $r = \frac{9}{5}$.
Substituting $r = \frac{9}{5}$ into $(1)$ gives $t = 2 + \frac{9}{5} = \frac{19}{5}$.

## Examples

### Example 1

Find the points of intersection of the following lines.

**d.** In $\mathbb{R}^3$: $l_1 : \vec{r} = (3,1,-1) + t\,(2,1,-3),\ t \in \mathbb{R}$ and $l_2 : \dfrac{x-7}{2} = \dfrac{y-12}{-4} = \dfrac{z-17}{5}$

**Solution**

**Method 1**

$$3t + 5r = -18 \qquad (3)$$

For the system of equations to be consistent (have a solution), $r = \frac{9}{5}$ and $t = \frac{19}{5}$ must also satisfy equation $(3)$.
Substituting, we get

$$3t + 5r = 3\left(\frac{19}{5}\right) + 5\left(\frac{9}{5}\right) = \frac{57+45}{5} = \frac{102}{5} \neq -18$$

Therefore, this system of equations has no solution and hence these two lines do not intersect.
By inspecting the parametric equations of both lines, we see that the direction vectors of the two lines are not scalar multiples of each other, so the lines are not parallel.
These lines are in $\mathbb{R}^3$, are not parallel, and do not intersect, and so $l_1$ and $l_2$ are skew lines.

## Examples

### Example 1

Find the points of intersection of the following lines.

**d.** In $\mathbb{R}^3$: $l_1 : \vec{r} = (3, 1, -1) + t\,(2, 1, -3),\ t \in \mathbb{R}$ and $l_2 : \dfrac{x-7}{2} = \dfrac{y-12}{-4} = \dfrac{z-17}{5}$

**Solution**

**Method 2**

As before, an alternative solution is to derive the parametric equations for $l_1$, and then substitute them into the symmetric equation for $l_2$ and verify consistency.

Substituting the parametric equations $x = 3 + 2t$ and $y = 1 + t$ of $l_1$ into the symmetric equations of $l_2$,

$$\begin{aligned} \frac{x-7}{2} &= \frac{y-12}{-4} \\ \frac{(3+2t)-7}{2} &= \frac{(1+t)-12}{-4} \\ \frac{2t-4}{2} &= \frac{t-11}{-4} \\ -4(t-2) &= t-11 \\ -5t &= -19 \\ t &= \frac{19}{5} \end{aligned}$$

Repeating the process by using the parametric equations $y = 1 + t$ and $z = -1 - 3t$, we get

$$\begin{aligned} \frac{y-12}{-4} &= \frac{z-17}{5} \\ \frac{(1+t)-12}{-4} &= \frac{(-1-3t)-17}{5} \\ \frac{t-11}{-4} &= \frac{-3t-18}{5} \\ 5t-55 &= 12t+72 \\ 7t &= -127 \\ t &= -\frac{127}{7} \neq \frac{19}{5} \end{aligned}$$

## Examples

### Example 1

Find the points of intersection of the following lines.

**d.** In $\mathbb{R}^3$: $l_1 : \vec{r} = (3, 1, -1) + t\,(2, 1, -3),\ t \in \mathbb{R}$ and $l_2 : \dfrac{x-7}{2} = \dfrac{y-12}{-4} = \dfrac{z-17}{5}$

**Solution**

Thus, there is no point on $l_1$ that also satisfies $l_2$.

There is no solution to this system of equations, so the lines do not intersect.