The Intersection of a Line and a Plane

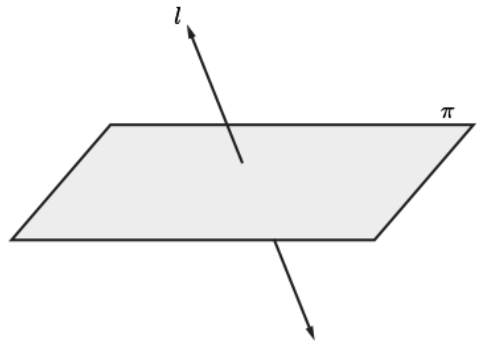
Introduction

In this fourth unit on vectors, we have learned three common ways to express the equation of a plane in $\mathbb{R}^3$. These are vector, parametric, and scalar equations of the plane.

In this module, we will consider how to determine if a given line intersects a given plane.

Given a line and a plane in $\mathbb{R}^3$, there are three possibilities for the intersection of the line with the plane.

1. The line and the plane intersect at a single point. There is exactly one solution.



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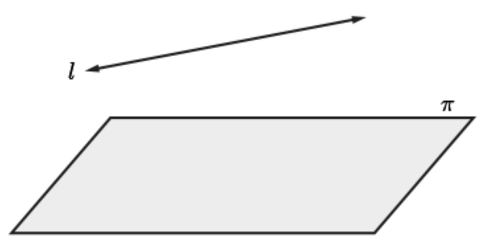
2. The line is parallel to the plane. The line and the plane do not intersect. There are no solutions.



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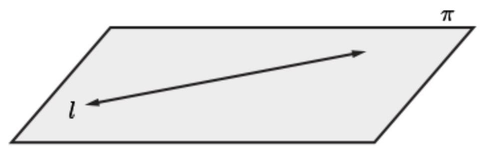
1. The line and the plane intersect at a single point. There is exactly one solution.
2. The line is parallel to the plane. The line and the plane do not intersect. There are no solutions.
3. The line lies on the plane, so every point on the line intersects the plane. There are an infinite number of solutions.



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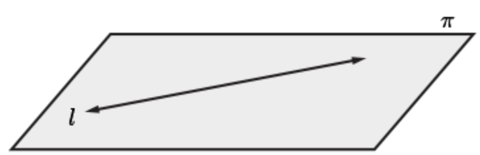

To determine algebraically whether or not a line intersects a plane, we may substitute the parametric equations of the line into the scalar equation of the plane.

The resulting linear equation reduces to one of three possibilities, with each determining the type of intersection between the line and the plane.

Examples

Example 1: A Single Point of Intersection

Determine all points of intersection between the line l described by

$$\begin{aligned} x &= 5 + 2t \\ y &= 1 + 6t \\ z &= -5t, \quad t \in \mathbb{R} \end{aligned}$$

and the plane $\pi : 2x + 3y + 5z - 1 = 0$.

Solution

Substitute the appropriate parametric equation of the line for the appropriate variable in the scalar equation of the plane.

$$\begin{aligned} 2(5 + 2t) + 3(1 + 6t) + 5(-5t) - 1 &= 0 \\ 10 + 4t + 3 + 18t - 25t - 1 &= 0 \\ -3t + 12 &= 0 \\ t &= 4 \end{aligned}$$

Substituting $t = 4$ into l gives $x = 13, y = 25$, and $z = -20$.

Therefore, the line intersects the plane at the point $(13, 25, -20)$.

Examples

Example 2: No Points of Intersection

Determine if the line l described by the symmetric equations

$$\frac{x-2}{5} = \frac{y+3}{-1} = \frac{z-1}{3}$$

intersects the plane $\pi : 3x - 5z = 6$.

Solution

Recall that we may set each expression of the symmetric equations equal to a parameter, t, and solve for the variables $x, y,$ and z to determine the parametric equations of the line.

$$\begin{aligned} \frac{x-2}{5} &= t \implies x = 2 + 5t \\ \frac{y+3}{-1} &= t \implies y = -3 - t \\ \frac{z-1}{3} &= t \implies z = 1 + 3t, \quad t \in \mathbb{R} \end{aligned}$$

Substituting the parametric equations into the scalar equation for π, we get

$$\begin{aligned} 3(2 + 5t) - 5(1 + 3t) &= 6 \\ 6 + 15t - 5 - 15t &= 6 \\ 0t &= 5 \end{aligned}$$

However, $0t \neq 5$ for all $t \in \mathbb{R}$ and so no value of t exists such that the line shares a point in common with the plane.

Thus, there is no point of intersection between the line and the plane.
This line is parallel to the plane and does not lie on the plane.

Examples

Example 2: No Points of Intersection

Determine if the line l described by the symmetric equations

$$\frac{x-2}{5} = \frac{y+3}{-1} = \frac{z-1}{3}$$

intersects the plane $\pi : 3x - 5z = 6$.

Solution

Recall we were just asked to determine if l intersects π.

We may also answer this question by quickly recognizing that the dot product of the direction of the line, $\vec{d} = (5, -1, 3)$, and the normal to the plane, $\vec{n} = (3, 0, -5)$, is equal to 0.
This means that these two vectors are orthogonal and so the line is parallel to the plane.

At this point, we must determine if the line lies on the plane.
If the line lies on the plane, then any point on the line satisfies the equation of the plane.

Substituting $(2, -3, 1)$ into π, we get $3(2) - 5(1) = 1 \neq 6$.
The point does not lie on the plane, so the line does not lie on the plane.

Therefore, there is no point of intersection between the line and the plane.

Examples

Example 3: An Infinite Number of Intersection Points

Determine all points of intersection between the line l described by the vector equation

$$\vec{r} = (8, -5, 0) + t(-2, 0, 1), t \in \mathbb{R}$$

and the plane having $x, y,$ and z-intercepts equal to $4, 5,$ and 2 respectively.

Solution

Written in parametric form, the vector equation of the line becomes

$$\begin{aligned} x &= 8 - 2t \\ y &= -5 \\ z &= t, \quad t \in \mathbb{R} \end{aligned}$$

Next, we determine the scalar equation of the plane. The plane passes through the points $X\ (4, 0, 0)$, $Y\ (0, 5, 0)$, and $Z\ (0, 0, 2)$, and so it has direction vectors $\overrightarrow{XY} = (-4, 5, 0)$ and $\overrightarrow{XZ} = (-4, 0, 2)$.

Since $(-4, 5, 0) \times (-4, 0, 2) = (10, 8, 20) = 2(5, 4, 10)$, then a normal vector to the plane is $\vec{n} = (5, 4, 10)$.
The plane has scalar equation of the form $5x + 4y + 10z + D = 0$ and substituting in $X\ (4, 0, 0)$ we get $D = -20$.

Therefore, the scalar equation of the plane is $5x + 4y + 10z - 20 = 0$.

Examples

Example 3: An Infinite Number of Intersection Points

Determine all points of intersection between the line l described by the vector equation

$$\vec{r} = (8, -5, 0) + t(-2, 0, 1), t \in \mathbb{R}$$

and the plane having x, y, and z-intercepts equal to $4, 5$, and 2 respectively.

Solution

Written in parametric form, the vector equation of the line is

$$\begin{aligned} x &= 8 - 2t \\ y &= -5 \\ z &= t, \quad t \in \mathbb{R} \end{aligned}$$

The scalar equation of the plane is $5x + 4y + 10z - 20 = 0$.

Substituting the parametric equations of l into the scalar equation of the plane, gives

$$\begin{aligned} 5(8 - 2t) + 4(-5) + 10(t) - 20 &= 0 \\ 40 - 10t - 20 + 10t - 20 &= 0 \\ 0t &= 0 \end{aligned}$$

This equation is true for any value of $t \in \mathbb{R}$. So, any point (x, y, z) on l lies on the plane, π.

Therefore, the line and the plane intersect in an infinite number of points satisfying the line $\vec{r} = (8, -5, 0) + t(-2, 0, 1)$.

Examples

Example 4

a. Show that the line $\vec{r} = (1, 0, 4) + t(-2, 2, 1), \quad t \in \mathbb{R}$ and the plane $6x + 2y + 3z - 12 = 0$ are not parallel.
b. Determine all points on this line $\vec{r} = (1, 0, 4) + t(-2, 2, 1)$ that are at a distance of 2 from the plane $6x + 2y + 3z - 12 = 0$.
c. Draw a sketch of the plane $6x + 2y + 3z - 12 = 0$.

Solution

a. The line has direction vector $\vec{d} = (-2, 2, 1)$ and $\vec{n} = (6, 2, 3)$ is the normal vector of the plane.

The line and the plane are parallel if and only if the direction vector of the line is orthogonal to the normal vector of the plane.
However, $(-2, 2, 1) \cdot (6, 2, 3) = -12 + 4 + 3 \neq 0$, and so they are not orthogonal.

Therefore, the line and the plane are not parallel.

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Solution

b. Recall last module when we derived the formula $d = \dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$ to determine the distance between a point (x_1, y_1, z_1) and a plane $Ax + By + Cz + D = 0$.

Thus, the distance d from the point (x_1, y_1, z_1) to the plane $6x + 2y + 3z - 12 = 0$ is

$$d = \frac{|6x_1 + 2y_1 + 3z_1 - 12|}{\sqrt{6^2 + 2^2 + 3^2}} = \frac{|6x_1 + 2y_1 + 3z_1 - 12|}{7}$$

The parametric equations of the given line are,

$$x = 1 - 2t, y = 2t, z = 4 + t$$

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The distance between any point on the line, $(x_1, y_1, z_1) = (1 - 2t, 2t, 4 + t)$, and the plane is given by

$$d = \frac{|6(1 - 2t) + 2(2t) + 3(4 + t) - 12|}{7} = \frac{1}{7}\,|6 - 5t|$$

When $d = 2$, we get $|6 - 5t| = 14$ and so $6 - 5t = 14$ or $6 - 5t = -14$, and so $t = -\dfrac{8}{5}$ or $t = 4$.

That is, there are two points on the line that are a distance of 2 from the plane.
When $t = -\dfrac{8}{5}$ we get the point $\left(\dfrac{21}{5}, -\dfrac{16}{5}, \dfrac{12}{5}\right)$, and when $t = 4$ we get the point $(-7, 8, 8)$.

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c. Draw a sketch of the plane $6x + 2y + 3z - 12 = 0$.

Solution

c. This final example is left as an exercise.

Can you determine the x-intercept, y-intercept, and z-intercept of this plane?
Can you use these three points to sketch the plane?