The Intersection of Three Planes

Introduction

Thus far, we have discussed the possible ways that two lines, a line and a plane, and two planes can intersect one another in 3-space.

Over the next two modules, we are going to look at the different ways that three planes can intersect in $\mathbb{R}^3$.

Examples

Example 1

Find all points of intersection of the following three planes:

$$\begin{aligned} x + 2y - 4z &= 3 \qquad (1)\\ -2x + y + 3z &= 4 \qquad (2)\\ 4x - 3y - z &= -2 \qquad (3) \end{aligned}$$

Solution

As we have done previously, we might begin with a quick look at the three normal vectors,
$\vec{n_1} = (1, 2, -4), \vec{n_2} = (-2, 1, 3)$, and $\vec{n_3} = (4, -3, -1)$.
Since no normal vector is parallel to another, we conclude that these three planes are non-parallel.

The approach we will take to finding points of intersection, is to eliminate variables until we can solve for one variable, and then substitute this value back into the previous equations to solve for the other two. This is known as **back substitution**.

First, we add 2 times equation (1) to equation (2) to eliminate x. This gives

$$\begin{array}{rr} 2\times & (x \ + 2y - 4z = 3) \\ + & -2x + \ y \ + 3z = \ 4 \\ \hline & 5y - 5z = 10 \end{array} \qquad (4)$$

Examples

Example 1

Find all points of intersection of the following three planes:

$$\begin{aligned} x + 2y - 4z &= 3 & (1)\\ -2x + y + 3z &= 4 & (2)\\ 4x - 3y - z &= -2 & (3) \end{aligned}$$

Solution

$$5y - 5z = 10 \qquad (4)$$

Now we use equations (1) and (3) to eliminate x again to produce another equation in y and z.
Adding -4 times (1) to (3), we get

$$\begin{array}{rrrrrrr} & -4\times(x + & 2y & - & 4z & = & 3) \\ + & 4x - & 3y & - & z & = & -2 \\ \hline & & -11y & + & 15z & = & -14 \end{array} \qquad (5)$$

We now use equations (4) and (5) to eliminate y and solve for z.
The solution we obtain for z will satisfy both equations (4) and (5), and thus satisfy equations (1), (2), and (3).

Examples

Example 1

Find all points of intersection of the following three planes:

$$\begin{aligned} x + 2y - 4z &= 3 & (1)\\ -2x + y + 3z &= 4 & (2)\\ 4x - 3y - z &= -2 & (3) \end{aligned}$$

Solution

Adding 11 times (4) to 5 times (5), we get

$$\begin{array}{rl} 11\times & (5y \quad - 5z \quad = 10) \\ +5\times & (-11y + 15z = -14) \\ \hline & \qquad 20z = 40 \end{array}$$

Solving, we get $z = 2$.

We back substitute $z = 2$ into equation (5) (or we could have used (4)), to get

$$\begin{aligned} -11y + 15z &= -14 \\ -11y + 15(2) &= -14 \\ -11y &= -44 \\ \therefore y &= 4 \end{aligned}$$

Examples

Example 1

Find all points of intersection of the following three planes:

$$\begin{aligned} x + 2y - 4z &= 3 \quad (1) \\ -2x + y + 3z &= 4 \quad (2) \\ 4x - 3y - z &= -2 \quad (3) \end{aligned}$$

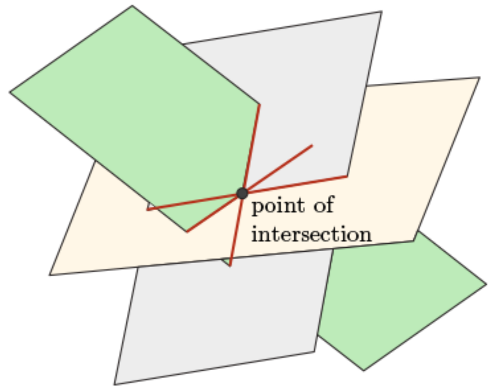
Solution

Substitute $y = 4, z = 2$ into any of (1), (2), or (3) to solve for x. Choosing (1), we get

$$\begin{aligned} x + 2y - 4z &= 3 \\ x + 2(4) - 4(2) &= 3 \\ \therefore x &= 3 \end{aligned}$$

Therefore, the solution to this system of three equations is $(x, y, z) = (3, 4, 2)$, a point.

This can be geometrically interpreted as three planes intersecting in a single point, as shown.



Introduction to Matrices

Mathematicians often develop new notation and ideas to help ease complicated calculations or procedures.
In the previous example (involving a linear system with multiple variables), the process of solving the system is simplified by using a matrix to help organize and eliminate variables efficiently.
Informally, a **matrix** (the plural form is matrices) is an array of m rows $\times$ n columns.
For example,

$$A = \begin{bmatrix} 1 & -2 & -\frac{2}{5} \\ \frac{7}{2} & -24 & 2 \end{bmatrix}$$

is a 2×3 matrix since it has 2 rows and 3 columns.
We often use a single, capital letter to represent a matrix, such as A in our example.
Further, A_{ij} is the notation used to reference the element in the i^{th} row and j^{th} column of matrix A. In this example, $A_{21} = \frac{7}{2}$.

Introduction to Matrices

To translate a system of linear equations into matrix form, we write the coefficients and the constant terms of the linear equations as elements in the corresponding locations in the matrix. From our first example earlier,

$$\begin{aligned} 1x + 2y - 4z &= 3 \\ -2x + 1y + 3z &= 4 \\ 4x - 3y - 1z &= -2 \end{aligned}$$

becomes

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 3 \\ -2 & 1 & 3 & 4 \\ 4 & -3 & -1 & -2 \end{array}\right]$$

Note that the matrix has a final column separated by a vertical line, which corresponds to the equal sign in the system of equations.

Such a matrix that includes the constant terms is known as an **augmented** matrix, and the elements in the matrix to the left of the vertical line form the **coefficient** matrix.

Introduction to Matrices

Gaussian Elimination

The benefit of using matrices becomes obvious when solving linear systems using a process known as **Gaussian elimination**.

In Gaussian elimination, the aim is to transform a system of equations in augmented matrix form, such as

$$A = \left[\begin{array}{ccc|c} 1 & 2 & -4 & 3 \\ -2 & 1 & 3 & 4 \\ 4 & -3 & -1 & -2 \end{array}\right]$$

into another augmented matrix of the form

$$D = \left[\begin{array}{ccc|c} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{array}\right]$$

where D is known as a matrix in **row echelon form**.

A matrix is said to be in row echelon form if

- all non-zero rows (rows with at least one non-zero element) are above any rows of all zeros, and
- the leading coefficient (the first non-zero element from the left) of a non-zero row is always strictly to the right of the leading coefficient of the row above it.

Introduction to Matrices

$$D = \left[\begin{array}{ccc|c} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{array}\right]$$

In the example of a 3×4 matrix, we are attempting to transform the elements lying below the main diagonal to zeros.

The **main diagonal** of an $m \times n$ matrix A $(m \leq n)$, is formed by the entries A_{ii} for $1 \leq i \leq m$.

For example, the entries $D_{11} = a$, $D_{22} = e$, and $D_{33} = h$ form the main diagonal of matrix D.

Introduction to Matrices

Row Operations

To transform the original augmented matrix into row echelon form, we perform **row operations**. Possible row operations are as follows:

- Multiplying each entry in one row by a (non-zero) scalar
- Adding (or subtracting) one row to (from) another
- A combination of the above two row operations
- Interchanging rows

Note that these row operations are identical to the operations permitted when solving systems of linear equations using the method of elimination.
When applied, each row operation creates an equivalent system of equations.
When an equivalent system is created, the solution to each set of equations remains the same.

Introduction to Matrices

Row Operations

Let us consider solving the system of linear equations from Example 1 again, this time using matrices.
We will perform the necessary row operations until the matrix is in row echelon form by following the same steps used when solving the system by the method of elimination.

$$\begin{array}{lrl} (1) & & x+2y-4z=3 \\ (2) & & -2x+y+3z=4 \\ (3) & & 4x-3y-z=-2 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & 2 & -4 & 3 \\ -2 & 1 & 3 & 4 \\ 4 & -3 & -1 & -2 \end{array}\right]$$

$$\begin{array}{lrl} (1) & & x+2y-4z=3 \\ (2) & 2\times(1)+(2)\rightarrow & 0x+5y-5z=10 \\ (3) & & 4x-3y-z=-2 \end{array} \qquad 2\times R_1+R_2\rightarrow \left[\begin{array}{rrr|r} 1 & 2 & -4 & 3 \\ 0 & 5 & -5 & 10 \\ 4 & -3 & -1 & -2 \end{array}\right]$$

$$\begin{array}{lrl} (1) & & x+2y-4z=3 \\ (2) & & 0x+5y-5z=10 \\ (3) & -4\times(1)+(3)\rightarrow & 0x-11y+15z=-14 \end{array} \qquad -4\times R_1+R_3\rightarrow \left[\begin{array}{rrr|r} 1 & 2 & -4 & 3 \\ 0 & 5 & -5 & 10 \\ 0 & -11 & 15 & -14 \end{array}\right]$$

$$\begin{array}{lrl} (1) & & x+2y-4z=3 \\ (2) & & 0x+5y-5z=10 \\ (3) & 11\times(2)+5\times(3)\rightarrow & 0x+0y+20z=40 \end{array} \qquad 11\times R_2+5\times R_3\rightarrow \left[\begin{array}{rrr|r} 1 & 2 & -4 & 3 \\ 0 & 5 & -5 & 10 \\ 0 & 0 & 20 & 40 \end{array}\right]$$

The matrix is now in row echelon form.

Introduction to Matrices

Row Operations

$$\left[\begin{array}{rrr|r} 1 & 2 & -4 & 3 \\ 0 & 5 & -5 & 10 \\ 0 & 0 & 20 & 40 \end{array}\right]$$

Considering the equations, we can now solve for the three variables by back substitution.

$$\begin{aligned} x+2y-4z&=3 && (1) \\ 5y-5z&=10 && (2) \\ 20z&=40 && (3) \end{aligned}$$

From (3), we get $z=2$.

Substituting $z=2$ into (2), we get $y=4$.
Finally, substituting $y=4, z=2$ into (1), we obtain $x=3$.

Therefore the solution is $(3,4,2)$, as we previously determined in Example 1.

Examples

Example 2

Use Gaussian elimination to determine all points of intersection of the following three planes:

$$\begin{aligned} 3x - 12y - 6z &= -9 \quad &(1) \\ -4x + y + 3z &= 1 \quad &(2) \\ -x + 4y + 2z &= 2 \quad &(3) \end{aligned}$$

Solution

Forming the augmented matrix, we proceed as follows:

$$\left[\begin{array}{ccc|c} 3 & -12 & -6 & -9 \\ -4 & 1 & 3 & 1 \\ -1 & 4 & 2 & 2 \end{array}\right]$$

$$\begin{array}{r} \\ 4R_1 + 3R_2 \to \\ R_1 + 3R_3 \to \end{array} \left[\begin{array}{ccc|c} 3 & -12 & -6 & -9 \\ 0 & -45 & -15 & -33 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

The last row of the matrix corresponds to the equation $0z = -3$, which has no solution.

Thus, this system of equations has no solution and therefore, the three corresponding planes have no points of intersection.

Geometric Interpretation

Algebraically, there is only one interpretation of a system of three equations with no solution - there is no point which satisfies all three equations simultaneously.
Recall that we call such a system of equations which has no solutions, **inconsistent**.

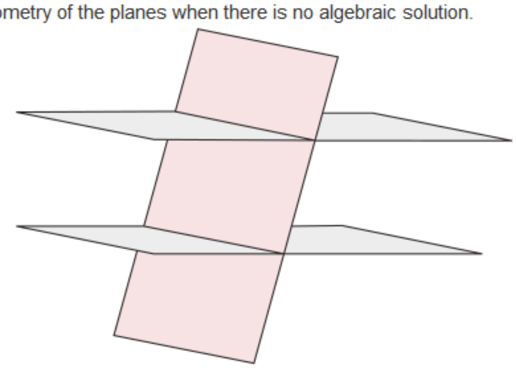
However, the algebraic interpretation does not give enough insight into the geometric interpretation.
In fact, there are multiple possibilities with regards to the geometry of the planes when there is no algebraic solution.

In Example 2, the system

$$\begin{array}{ll} (1) & 3x - 12y - 6z = -9 \\ (2) & -4x + y + 3z = 1 \\ (3) & -x + 4y + 2z = 2 \end{array}$$

had no solution.
To visualize this geometrically, we note that equations (1) and (3) have normal vectors $\overrightarrow{n_1} = (3, -12, -6)$ and $\overrightarrow{n_3} = (-1, 4, 2)$, and so $\overrightarrow{n_1} = -3\overrightarrow{n_3}$.

Thus, the planes described by (1) and (3) are parallel, but distinct since $-9 \neq -3(2)$.



The normal vector of the second plane, $\overrightarrow{n_2} = (-4, 1, 3)$ is not parallel to either of these, so the second plane must intersect each of the other two planes in a line.

This situation is drawn here:

Closing Thoughts

In the next module, we will consider other possible ways that three planes can intersect, including those in which the solution contains a parameter.

In this module, we considered only two of the eight distinct possible types of intersection.

Can you visualize the other six? Can you draw the other six cases?

Attempt this on your own and then open the investigation within this module to check your work.