The Intersection of Two Planes

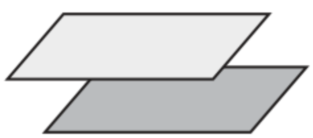
Recall

Recall that, given a line and a plane in $\mathbb{R}^3$, there are three possibilities for the intersection of the line with the plane.

- The line is parallel to the plane (orthogonal to a normal vector of the plane). The line and the plane do not intersect. There are no solutions.
- The line and the plane intersect at a single point. There is exactly one solution.
- The line lies in the plane, so every point on the line intersects the plane. There are an infinite number of solutions.

The Intersection of Two Planes

Similarly, there are also three possibilities for the intersection of two planes:



The two planes are parallel and distinct.

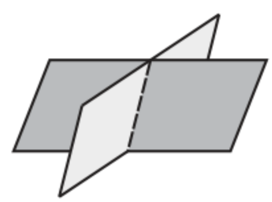
The normal vectors of the planes are scalar multiples of each other, but the equations of the planes in scalar form are not multiples of each other.

There are no points that satisfy the equations of both planes.

The two planes are coincident.

The normal vectors of the planes are scalar multiples of each other, and the equations in scalar form are multiples of each other.

Any point on either plane satisfies the equation of the other plane.



The two planes intersect in a line.

The normal vectors of the planes are not scalar multiples of each other.

Solving the system of two equations (the equations of the two planes) in three variables will give the equation of the line of intersection.

Examples

Example 1

Determine the intersection of the two planes:

$$\begin{aligned}\pi_1: &\quad 3x-2y+4z-17=0\\ \pi_2: &\quad (x,y,z)=(5,-1,0)+t(2,1,-1)+s(-10,-3,6),\ s,\ t\in\mathbb{R}\end{aligned}$$

Solution

A normal vector of π_1 is $\overrightarrow{n_1}=(3,-2,4)$.
From π_2, $\overrightarrow{d_2}=(2,1,-1)$ and $\overrightarrow{e_2}=(-10,-3,6)$ are two direction vectors for the plane.
It follows that a normal vector for π_2 is $\overrightarrow{n_2}=\overrightarrow{d_2}\times\overrightarrow{e_2}=(3,-2,4)$.

Since the two normal vectors are scalar multiples of each other (in fact, they are equal), the planes are parallel.
The point $P_2\,(5,-1,0)$ lies on π_2.
To determine if the planes are coincident or if they are parallel and distinct, we substitute $P_2\,(5,-1,0)$ into the equation for $\pi_1: 3x-2y+4z-17=0$.
Doing this, we get

$$\begin{aligned}3x-2y+4z-17&=3(5)-2(-1)+4(0)-17\\&=17-17\\&=0\end{aligned}$$

Thus, P_2 satisfies the equation of π_1 and thus lies on both π_1 and π_2.

Examples

Example 1

Determine the intersection of the two planes:

$$\begin{aligned}\pi_1: &\quad 3x-2y+4z-17=0\\ \pi_2: &\quad (x,y,z)=(5,-1,0)+t(2,1,-1)+s(-10,-3,6),\ s,\ t\in\mathbb{R}\end{aligned}$$

Solution

We have shown that the two planes are parallel and have a point in common, so the planes are in fact coincident.
The solution is the set of all points in either plane.

Note:

If $P_2\,(5,-1,0)$ did not lie on π_1, the planes would have been parallel and distinct.
In this case, there would be no intersection between the two planes.

Examples

Example 1

Determine the intersection of the two planes:

$$\begin{aligned} \pi_1: &\quad 3x - 2y + 4z - 17 = 0 \\ \pi_2: &\quad (x, y, z) = (5, -1, 0) + t(2, 1, -1) + s(-10, -3, 6),\ s,\ t \in \mathbb{R} \end{aligned}$$

Solution

Alternative Method

We begin by rewriting the equation of π_2 in parametric form.

$\pi_2 : (x, y, z) = (5, -1, 0) + t(2, 1, -1) + s(-10, -3, 6)$ in parametric form is

$$x = 5 + 2t - 10s,\ y = -1 + t - 3s,\ z = -t + 6s$$

Substituting (x, y, z) (that is, π_2) into $\pi_1 : 3x - 2y + 4z - 17 = 0$, we get

$$\begin{aligned} 3(5 + 2t - 10s) - 2(-1 + t - 3s) + 4(-t + 6s) - 17 &= 0 \\ 15 + 6t - 30s + 2 - 2t + 6s - 4t + 24s - 17 &= 0 \\ 6t - 6t - 30s + 30s + 17 - 17 &= 0 \\ \therefore 0 &= 0 \end{aligned}$$

This final equation is true for all values of the parameters t and s.

Thus, every point (x, y, z) on π_2 satisfies the equation of π_1, so the planes are coincident.

Examples

Example 1

Determine the intersection of the two planes:

$$\begin{aligned} \pi_1: &\quad 3x - 2y + 4z - 17 = 0 \\ \pi_2: &\quad (x, y, z) = (5, -1, 0) + t(2, 1, -1) + s(-10, -3, 6),\ s,\ t \in \mathbb{R} \end{aligned}$$

Solution

Alternative Method

In the method just used, how would the result of simplifying the equation in s and t differ if the two planes were parallel and distinct?

What if the two planes intersected in a line?

Examples

Example 2

Find the intersection of the two planes:

$$\begin{aligned} \pi_1: &\quad x - y - z - 12 = 0 \\ \pi_2: &\quad 3x - 2y - 4z - 8 = 0 \end{aligned}$$

Solution

Normal vectors for the planes are $\overrightarrow{n_1} = (1, -1, -1)$ and $\overrightarrow{n_2} = (3, -2, -4)$ for π_1 and π_2, respectively.
By inspection, the normal vectors are not scalar multiples of each other, so the two planes are not parallel and must intersect in a line.
Together, the equations of the two planes give a linear system of two equations in three variables.
To determine the equation of the line of intersection of these two planes, we solve this system of equations.

$$\begin{array}{rcrcrcrcrcr} & -3\pi_1: & -3x & + & 3y & + & 3z & + & 36 & = & 0 \\ + & \pi_2: & 3x & - & 2y & - & 4z & - & 8 & = & 0 \\ \hline & & & & y & - & z & + & 28 & = & 0 \end{array}$$

Rearranging this equation, we get $y = z - 28$.
Since the solution to $y = z - 28$ *depends* on z, and z has no constraints, we choose z to be the parameter $t,\ t \in \mathbb{R}$.
Thus, $y = t - 28$.

Examples

Example 2

Find the intersection of the two planes:

$$\begin{aligned} \pi_1: &\quad x - y - z - 12 = 0 \\ \pi_2: &\quad 3x - 2y - 4z - 8 = 0 \end{aligned}$$

Solution

$$y = t - 28,\ z = t$$

Substituting for y and z in the equation for π_1, we get

$$\begin{aligned} x - y - z - 12 &= 0 \\ x - (t - 28) - (t) - 12 &= 0 \\ \therefore x &= 2t - 16 \end{aligned}$$

The intersection of the two planes is the line $x = 2t - 16,\ y = t - 28,\ z = t,\ t \in \mathbb{R}$.

This system of equations was *dependent* on one of the variables (we chose z in our solution).
Any system of equations in which some variables are each dependent on one or more of the other remaining variables is called a **dependent** system.
Conversely, the system is called **independent**.

Examples

Example 3

Find the intersection of the two planes:

$$\begin{aligned} \pi_1: &\quad 3x - 4z + 6 = 0 \\ \pi_2: &\quad x + y + 5z - 5 = 0 \end{aligned}$$

Solution

Normal vectors for the planes are $\overrightarrow{n_1} = (3, 0, -4)$ and $\overrightarrow{n_2} = (1, 1, 5)$ for π_1 and π_2, respectively.

By inspection, the normal vectors are not scalar multiples of each other, so the two planes are not parallel and must intersect in a line.

Since the equation of π_1 has only two variables, we can rearrange the equation to get $3x = 4z - 6$ or $x = \frac{4}{3}z - 2$.

Letting $z = 3t,\ t \in \mathbb{R}$, we get $x = \frac{4}{3}(3t) - 2$ and so $x = 4t - 2$.

Substituting x and z into the equation for π_2, we get $(4t - 2) + y + 5(3t) - 5 = 0$ or $y = -19t + 7$.

The intersection of the two planes is the line $x = 4t - 2,\ y = -19t + 7,\ z = 3t,\ t \in \mathbb{R}$.

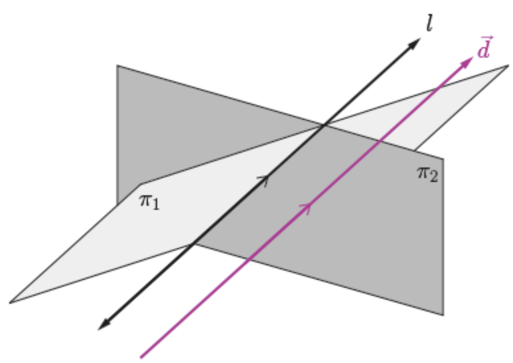
Examples

Example 4

Find the intersection of the two planes:

$$\begin{aligned} \pi_1: &\quad 3x - 4z + 6 = 0 \\ \pi_2: &\quad x + y + 5z - 5 = 0 \end{aligned}$$

Use a different method from that used in example 3.



Solution

Normal vectors for the planes are $\overrightarrow{n_1} = (3, 0, -4)$ and $\overrightarrow{n_2} = (1, 1, 5)$ for π_1 and π_2, respectively.

By inspection, the normal vectors are not scalar multiples of each other, so the two planes are not parallel and must intersect in a line.

Since the line of intersection, l, lies on π_1, it is orthogonal to the normal vector $\overrightarrow{n_1} = (3, 0, -4)$ of this plane.

Similarly, since l lies on π_2, it is orthogonal to the normal vector $\overrightarrow{n_2} = (1, 1, 5)$ of this plane.

Let $\vec{d} = \overrightarrow{n_1} \times \overrightarrow{n_2} = (4, -19, 3)$.

This vector, $\vec{d}$, is orthogonal to both normal vectors and is therefore parallel to the line of intersection.

So, $\vec{d}$ is a direction vector for the line of intersection.

Examples

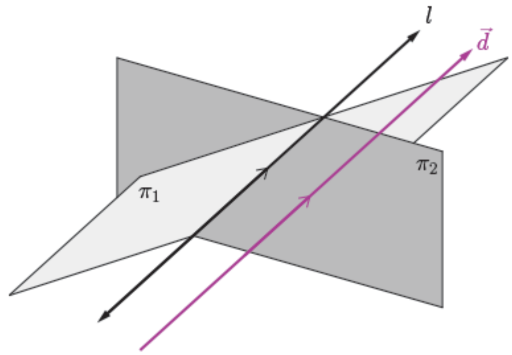
Example 4

Find the intersection of the two planes:

$$\begin{aligned} \pi_1: &\quad 3x - 4z + 6 = 0 \\ \pi_2: &\quad x + y + 5z - 5 = 0 \end{aligned}$$

Use a different method from that used in example 3.

Solution



Next, we find a point on this line of intersection.
Let $z = 0$ and solve the system of equations
($3x + 6 = 0$ and $x + y - 5 = 0$)
to get $x = -2$ and $y = 7$.
So, the point $(-2, 7, 0)$ lies on both planes and therefore it lies on the line of intersection.

Thus, the intersection of the two planes is the line having direction $\vec{d} = (4, -19, 3)$ and passing through the point $(-2, 7, 0)$; that is,

$$x = 4t - 2,\ y = -19t + 7,\ z = 3t,\ t \in \mathbb{R}$$

as we previously found in example 3.