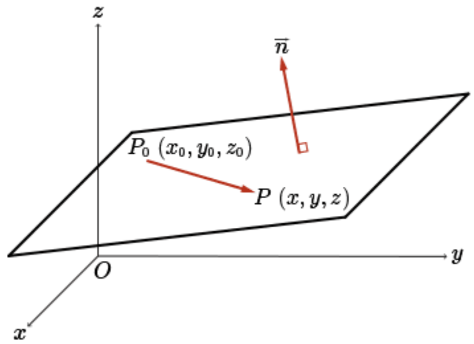
The Scalar Equation of a Plane

Introduction

In $\mathbb{R}^2$, the scalar (Cartesian) equation of a line was derived using the notion of a normal vector, $\vec{n}$, and a vector in the plane, $\overrightarrow{P_0P}$.
This notion can be extended to three dimensions to derive the scalar equation of a plane.

Let $\vec{n} = (A, B, C)$ be a normal vector to a plane that contains the fixed point $P_0\ (x_0, y_0, z_0)$.
Let $P\ (x, y, z)$ be any other arbitrary point in the plane. Then by the definition of the dot product,



$$\begin{aligned} \vec{n} \cdot \overrightarrow{P_0P} &= 0 \\ (A, B, C) \cdot (x - x_0, y - y_0, z - z_0) &= 0 \\ A(x - x_0) + B(y - y_0) + C(z - z_0) &= 0 \\ Ax - Ax_0 + By - By_0 + Cz - Cz_0 &= 0 \\ Ax + By + Cz + \underbrace{(-Ax_0 - By_0 - Cz_0)}_{\text{constant } D} &= 0 \\ \therefore Ax + By + Cz + D &= 0 \end{aligned}$$

is the scalar equation of a plane in $\mathbb{R}^3$.

In particular, all points in the plane must satisfy this equation and conversely, any point which satisfies this equation must lie on the plane.
Observe, as we did for the scalar equation of a line in $\mathbb{R}^2$, that the components of the normal vector (A, B, C) form the coefficients of the scalar equation (and vice versa).

Examples

Example 1

Find the scalar equation of a plane with normal $(1, -2, 5)$ and containing the point $P_0\ (3, 4, -1)$.

Solution

Let $P\ (x, y, z)$ be any point in the plane. Then,

$$\begin{aligned} \vec{n} \cdot \overrightarrow{P_0P} &= 0 \\ (1, -2, 5) \cdot (x - 3, y - 4, z + 1) &= 0 \\ 1(x - 3) - 2(y - 4) + 5(z + 1) &= 0 \\ x - 3 - 2y + 8 + 5z + 5 &= 0 \\ x - 2y + 5z + 10 &= 0 \end{aligned}$$

is the scalar equation of the plane.

Alternative Method

Since $\vec{n} = (1, -2, 5)$ is the normal, the scalar equation of the plane is of the form $x - 2y + 5z + D = 0$, with the constant, D, to be determined.
Since the plane passes through the point P_0, then P_0 satisfies the equation of the plane. Substituting $P_0\ (3, 4, -1)$ into this equation, we get

$$\begin{aligned} 3 - 2(4) + 5(-1) + D &= 0 \\ -10 + D &= 0 \\ D &= 10 \end{aligned}$$

Therefore, the scalar equation of the plane is $x - 2y + 5z + 10 = 0$.

Examples

Example 2

Find the scalar equation of the plane containing the line $(x, y, z) = (-4, 0, 5) + t(3, 1, 2)$ and the point $(1, 1, 0)$.

Solution

Using the direction vector of the line, one vector in the plane is $(3, 1, 2)$.
The vector extending from $(-4, 0, 5)$ to $(1, 1, 0)$ is another vector in the plane (since both points lie in the plane).
That is, $(1, 1, 0) - (-4, 0, 5) = (5, 1, -5)$ is a second direction vector for the plane.
By inspection, these two vectors are not collinear.

The cross product of these two vectors gives a vector that is perpendicular to both, and hence perpendicular to the plane (that is, a normal to the plane).

$$\begin{aligned}\vec{n} &= (3, 1, 2) \times (5, 1, -5)\\ &= (-7, 25, -2)\end{aligned}$$

Thus, the scalar equation of the plane is of the form $-7x + 25y - 2z + D = 0$.

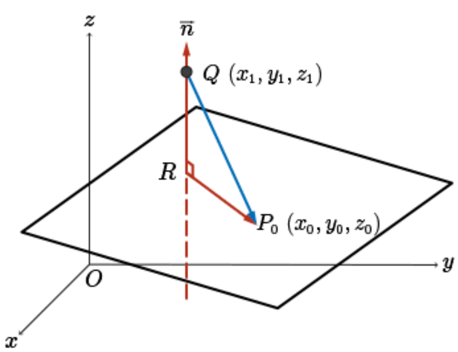
Substituting the point $(1, 1, 0)$, we get

$$\begin{aligned}-7(1) + 25(1) - 2(0) + D &= 0\\ 18 + D &= 0\\ D &= -18\end{aligned}$$

The scalar equation of the plane is $-7x + 25y - 2z - 18 = 0$ or $7x - 25y + 2z + 18 = 0$.

The Distance Between a Point and a Plane

The distance from a point $Q\ (x_1, y_1, z_1)$ to a plane $Ax + By + Cz + D = 0$ in $\mathbb{R}^3$ is defined as the **shortest** geometric distance.
If a line through Q and perpendicular to the plane intersects the plane at R, then the shortest distance is $|\overrightarrow{QR}|$.
Let $P_0\ (x_0, y_0, z_0)$ be a point in the plane.
Then the distance from Q to R is the magnitude of the projection (the scalar projection) of $\overrightarrow{QP_0}$ onto the normal $\vec{n} = (A, B, C)$.



$$\begin{aligned}|\overrightarrow{QR}| = |\textbf{proj}\ (\overrightarrow{QP_0}\ \text{onto}\ \vec{n})| &= \frac{|\overrightarrow{QP_0} \cdot \vec{n}|}{|\vec{n}|}\\ &= \frac{|(x_0 - x_1, y_0 - y_1, z_0 - z_1) \cdot (A, B, C)|}{\sqrt{A^2 + B^2 + C^2}}\\ &= \frac{|Ax_0 - Ax_1 + By_0 - By_1 + Cz_0 - Cz_1|}{\sqrt{A^2 + B^2 + C^2}}\\ &= \frac{|-Ax_1 - By_1 - Cz_1 + (Ax_0 + By_0 + Cz_0)|}{\sqrt{A^2 + B^2 + C^2}}\\ &= \frac{|-(Ax_1 + By_1 + Cz_1) + (Ax_0 + By_0 + Cz_0)|}{\sqrt{A^2 + B^2 + C^2}}\end{aligned}$$

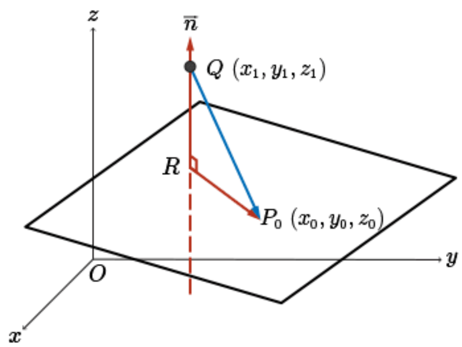
The Distance Between a Point and a Plane

Since the point $P\ (x_0, y_0, z_0)$ lies in the plane, then it satisfies the equation of the plane.
That is, $Ax_0 + By_0 + Cz_0 + D = 0$ and so

$$Ax_0 + By_0 + Cz_0 = -D$$

Substituting this into the previous equation, we get

$$\begin{aligned}|\overrightarrow{QR}| &= \frac{|-(Ax_1 + By_1 + Cz_1) + (Ax_0 + By_0 + Cz_0)|}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{|-(Ax_1 + By_1 + Cz_1) - D|}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{|-(Ax_1 + By_1 + Cz_1 + D)|}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}\end{aligned}$$



A similar process can be used to find a formula for the distance from a point (x_1, y_1) to a line $Ax + By + C = 0$ in $\mathbb{R}^2$.

That is, $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.

Examples

Example 3

Find the distance from the point $Q\ (1, 4, 7)$ to the plane containing the points $X\ (0, 4, -1)$, $Y\ (6, 2, 5)$, and $Z\ (3, -1, 6)$.

Solution

First, determine the scalar equation of the plane by using the three points to generate two vectors, $\vec{d_1}$ and $\vec{d_2}$, followed by the normal vector.

$$\begin{aligned}\overrightarrow{XY} = (6, -2, 6) = 2(3, -1, 3) &\implies \vec{d_1} = (3, -1, 3) \\ \overrightarrow{XZ} = (3, -5, 7) &\implies \vec{d_2} = (3, -5, 7) \\ (3, -1, 3) \times (3, -5, 7) = (8, -12, -12) = 4(2, -3, -3) &\implies \vec{n} = (2, -3, -3)\end{aligned}$$

Therefore, the equation of the plane is of the form $2x - 3y - 3z + D = 0$.
Substituting $X\ (0, 4, -1)$ to determine D, we get

$$\begin{aligned}2(0) - 3(4) - 3(-1) + D &= 0 \\ D &= 9\end{aligned}$$

Therefore, the equation of the plane is $2x - 3y - 3z + 9 = 0$.
Finally, the distance from the point $Q\ (1, 4, 7)$ to the plane $2x - 3y - 3z + 9 = 0$ is

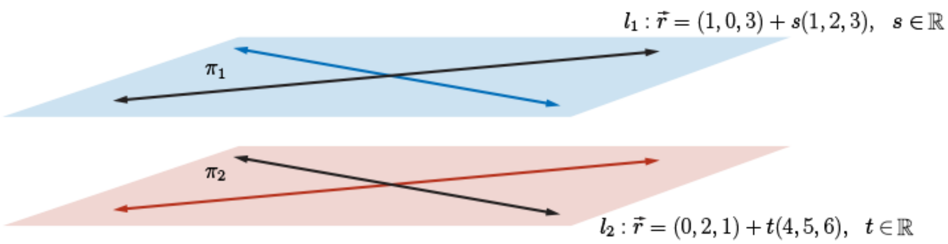
$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|2(1) - 3(4) - 3(7) + 9|}{\sqrt{(2)^2 + (-3)^2 + (-3)^2}} = \frac{|-22|}{\sqrt{22}} = \sqrt{22}$$

Examples

Example 4: The Distance Between Skew Lines

Determine the distance between the two lines $l_1 : \vec{r} = (1,0,3) + s(1,2,3), \quad s \in \mathbb{R}$ and $l_2 : \vec{r} = (0,2,1) + t(4,5,6), \quad t \in \mathbb{R}$.

Solution



First, you should verify that these are indeed skew lines, that is, they are not parallel and do not intersect. (For a refresher on how to do this, look back at the module titled "The Intersection of Two Lines.")

One method to find the distance between skew lines involves determining the equations of two planes that are parallel to one another.

Let the first plane, π_1, contain the line l_1 along with a second line with direction $\vec{d_2} = (4,5,6)$, the direction of l_2.

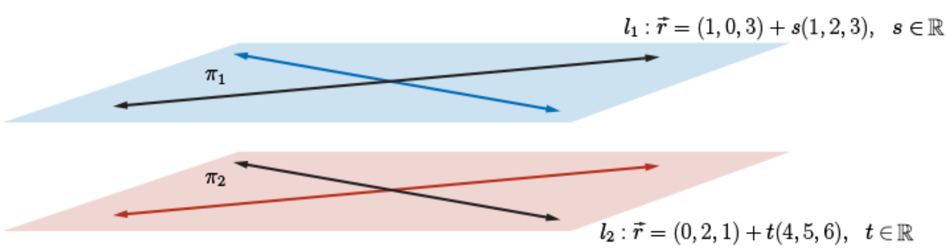
Similarly, let the second plane, π_2, contain the line l_2 along with a second line with direction $\vec{d_1} = (1,2,3)$, the direction of l_1.

Examples

Example 4: The Distance Between Skew Lines

Determine the distance between the two lines $l_1 : \vec{r} = (1,0,3) + s(1,2,3), \quad s \in \mathbb{R}$ and $l_2 : \vec{r} = (0,2,1) + t(4,5,6), \quad t \in \mathbb{R}$.

Solution



Determine the equation of π_1:

This plane has direction vectors $\vec{d_1} = (1,2,3)$ and $\vec{d_2} = (4,5,6)$ and since $\vec{n} = (1,2,3) \times (4,5,6) = (-3,6,-3) = -3(1,-2,1)$ then the normal is $\vec{n} = (1,-2,1)$.

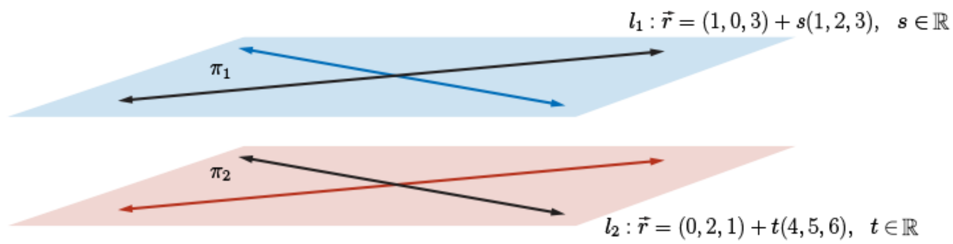
Substituting $(1,0,3)$ into $x - 2y + z + D = 0$ gives $D = -4$, and so the equation of π_1 is $x - 2y + z - 4 = 0$.

Examples

Example 4: The Distance Between Skew Lines

Determine the distance between the two lines $l_1 : \vec{r} = (1, 0, 3) + s(1, 2, 3), \quad s \in \mathbb{R}$ and $l_2 : \vec{r} = (0, 2, 1) + t(4, 5, 6), \quad t \in \mathbb{R}$.

Solution



Determine the equation of π_2:

This plane has the same two direction vectors $\vec{d_1} = (1, 2, 3)$ and $\vec{d_2} = (4, 5, 6)$, and so $\vec{n} = (1, -2, 1)$.

Substituting $(0, 2, 1)$ into $x - 2y + z + D = 0$ gives $D = 3$, and so the equation of π_2 is $x - 2y + z + 3 = 0$.

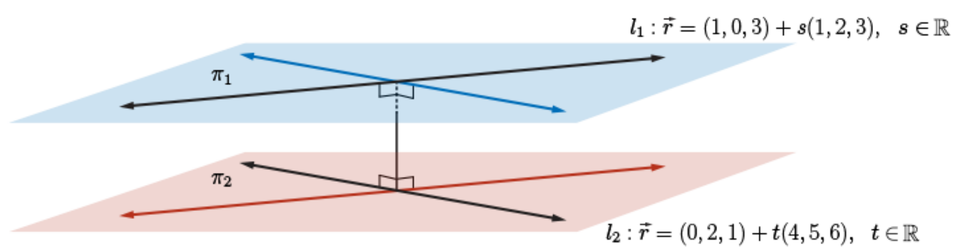
Since the two direction vectors used to help define π_1 are identical to the two direction vectors for π_2, then π_1 and π_2 are parallel planes.

Examples

Example 4: The Distance Between Skew Lines

Determine the distance between the two lines $l_1 : \vec{r} = (1, 0, 3) + s(1, 2, 3), \quad s \in \mathbb{R}$ and $l_2 : \vec{r} = (0, 2, 1) + t(4, 5, 6), \quad t \in \mathbb{R}$.

Solution



The original problem of finding the distance between the skew lines, l_1 and l_2, has now been reduced to finding the distance between these two parallel planes, π_1 and π_2.
The distance between the two planes is equal to the distance between any point on π_1 and the second plane.
Choose $Q\ (1, 0, 3)$ on π_1.
The distance between $Q\ (1, 0, 3)$ and $\pi_2 : x - 2y + z + 3 = 0$ is

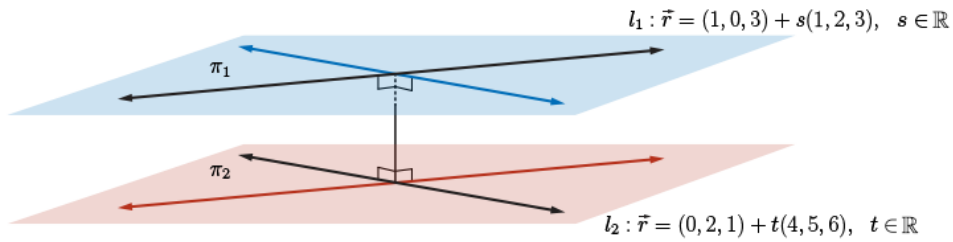
$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|1(1) - 2(0) + 1(3) + 3|}{\sqrt{(1)^2 + (-2)^2 + (1)^2}} = \frac{7}{\sqrt{6}} = \frac{7}{6}\sqrt{6}$$

Examples

Example 4: The Distance Between Skew Lines

Determine the distance between the two lines $l_1 : \vec{r} = (1,0,3) + s(1,2,3), \quad s \in \mathbb{R}$ and $l_2 : \vec{r} = (0,2,1) + t(4,5,6), \quad t \in \mathbb{R}$.

Solution



Another possible method for determining the distance between skew lines is to find the point A on l_1 and the point B on l_2 such that the distance AB is as small as possible.

To determine the coordinates of these two points, we need to use the fact that vector $\overrightarrow{AB}$ is orthogonal to the direction vector of each line.

Following this method through, using the lines given in the previous example, would be an excellent exercise for you to try.