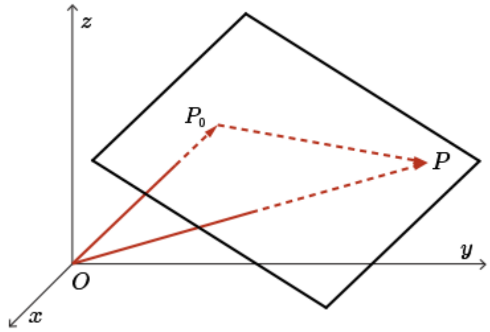
Vector and Parametric Equations of a Plane

Introduction

A line (1-dimensional object) in space may be described using a point on the line and a **single** direction vector.

It naturally follows that a plane (2-dimensional object) can be described using a point in the plane and **two** direction vectors. However, recall that two collinear vectors lie on (and describe) the same line; thus, the two direction vectors describing the plane must be **non**-collinear.

The vector equation of a plane describes the position vector, $\overrightarrow{OP}$, of any arbitrary point, P, in the plane.
To derive the vector equation for an arbitrary plane, consider the following:



Let P_0 be a fixed point in the plane, and let P be any other arbitrary point in the plane.
The coordinates of P can be determined from the vector sum

$$(1) \qquad \overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$$

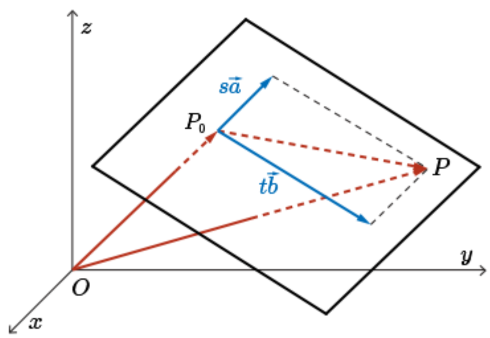
as shown in the diagram.

Introduction

If we take any two non-collinear vectors in the plane, then any arbitrary vector in the plane (such as $\overrightarrow{P_0P}$) can be expressed as a vector sum of scalar multiples of these two vectors.

Let $\vec{a}$ and $\vec{b}$ be two such vectors in the plane.
Then $s\vec{a} + t\vec{b} = \overrightarrow{P_0P}$, for some $s, t \in \mathbb{R}$ as shown in the diagram.
This vector $s\vec{a} + t\vec{b}$ is called a linear combination of vectors $\vec{a}$ and $\vec{b}$.

Substituting this into equation (1), $\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$, we get $\overrightarrow{OP} = \overrightarrow{OP_0} + s\vec{a} + t\vec{b}, \quad s, t \in \mathbb{R}$ which is the vector equation of the plane.



The **vector equation of a plane**, $\overrightarrow{OP} = \overrightarrow{OP_0} + s\vec{a} + t\vec{b}$, gives the position vector $\overrightarrow{OP}$ of any point $P\ (x, y, z)$ in the plane. It is written as the sum of the position vector $\overrightarrow{OP_0}$ of any fixed point $P_0\ (x_0, y_0, z_0)$ in the plane and a linear combination of any two non-collinear vectors, $\vec{a}$ and $\vec{b}$, that lie in the plane.
An alternative method of writing the vector equation is to let $\vec{r} = \overrightarrow{OP}$ and $\vec{r_0} = \overrightarrow{OP_0}$, giving

$$\vec{r} = \vec{r_0} + s\vec{a} + t\vec{b}, \ s, t \in \mathbb{R}$$

Parametric Equations of a Plane

Rewriting the vector equation of a plane into its x, y, and z components, we get

$$\vec{r} = \vec{r_0} + s\vec{a} + t\vec{b}, \quad s,t \in \mathbb{R}$$
$$(x,y,z) = (x_0,y_0,z_0) + s(a_1,a_2,a_3) + t(b_1,b_2,b_3), \quad s,t \in \mathbb{R}$$

We then see that the **parametric equations of a plane** are

$$x = x_0 + sa_1 + tb_1$$
$$y = y_0 + sa_2 + tb_2$$
$$z = z_0 + sa_3 + tb_3, \quad s,t \in \mathbb{R}$$

Examples

Example 1

Points $U\ (3,0,-1), V\ (-3,1,2)$, and $W\ (4,7,-1)$ lie in a plane. Find the vector and parametric equations of the plane.

Solution

The vector equation of a plane requires a point in the plane and two non-collinear vectors. Observe that $\overrightarrow{UV} = (-6,1,3)$ and $\overrightarrow{UW} = (1,7,0)$ are non-collinear.

We can use the position vector of any of the three points U, V, or W as $\vec{r_0}$.

Choosing $U\ (3,0,-1)$ gives the vector equation of the plane as

$$\vec{r} = (3,0,-1) + s(-6,1,3) + t(1,7,0), \quad s,t \in \mathbb{R}$$

from which the parametric equations are

Example 2

Does $(-1,11,2)$ lie in the plane described by $\vec{r} = (-6,6,-1) + s(3,4,0) + t(8,-1,-3), s,t \in \mathbb{R}$?

Examples

Example 2

Does $(-1, 11, 2)$ lie in the plane described by $\vec{r} = (-6, 6, -1) + s(3, 4, 0) + t(8, -1, -3), s, t \in \mathbb{R}$?

Solution

If the point lies in the plane, its coordinates must satisfy the parametric equations.

$$\begin{aligned} (1) \quad & -1 = -6 + 3s + 8t \\ (2) \quad & 11 = 6 + 4s - t \\ (3) \quad & 2 = -1 - 3t \end{aligned}$$

From equation (3), we solve to get $t = -1$. Substituting $t = -1$ into (2) gives

$$\begin{aligned} 11 &= 6 + 4s + 1 \\ 4 &= 4s \\ s &= 1 \end{aligned}$$

So, $s = 1$ and $t = -1$. Finally, we check if these values for the parameters satisfy equation (1).

Since $-6 + 3s + 8t = -6 + 3(1) + 8(-1) = -11 \neq -1$, then $(-1, 11, 2)$ does not lie in the plane.

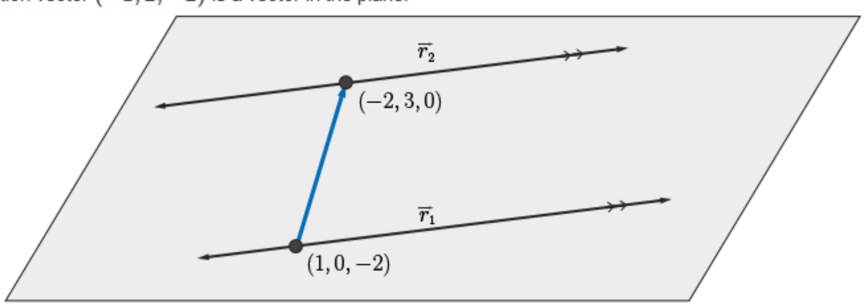
Examples

Example 3

Two parallel (and distinct) lines $\vec{r_1} = (1, 0, -2) + s(2, -4, 4), \quad s \in \mathbb{R}$ and $\vec{r_2} = (-2, 3, 0) + t(-1, 2, -2), \quad t \in \mathbb{R}$ lie in a plane. Find the equation of the plane.

Solution

The direction vector $(-1, 2, -2)$ is a vector in the plane.



To obtain a second non-collinear vector in the plane, we determine the vector between the two given points on the lines; this is the vector $(-2, 3, 0) - (1, 0, -2) = (-3, 3, 2)$.

Choosing $(1, 0, -2)$ as a fixed point in the plane, we get a vector equation for the plane to be

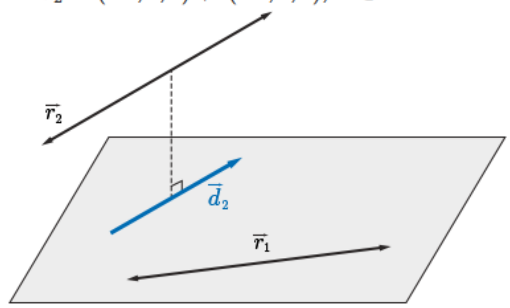
$$\vec{r} = (1, 0, -2) + c(-1, 2, -2) + k(-3, 3, 2), \quad c, k \in \mathbb{R}$$

Examples

Example 4

Determine the vector and parametric equations of the plane that contains the line $\vec{r_1} = (3,5,-1) + s(1,1,2), \quad s \in \mathbb{R}$ and is parallel to the line $\vec{r_2} = (-2,0,4) + t(-2,1,8), \quad t \in \mathbb{R}$.

Solution



Since the plane contains the line $\vec{r_1} = (3,5,-1) + s(1,1,2)$, then the point $(3,5,-1)$ lies on the plane and $\vec{d_1} = (1,1,2)$ is a direction vector for the plane.

The plane is also parallel to the line $\vec{r_2} = (-2,0,4) + t(-2,1,8)$, and so $\vec{d_2} = (-2,1,8)$ is a second direction vector for the plane.

Since $(-2,1,8)$ is not a scalar multiple of $(1,1,2)$, the direction vectors are not parallel.

Therefore, the vector equation of the plane is

$$\vec{r} = (3,5,-1) + c(-2,1,8) + k(1,1,2), \quad c,k \in \mathbb{R}$$

and the parametric equations for the plane are

$$\begin{aligned} x &= 3 - 2c + k \\ y &= 5 + c + k \\ z &= -1 + 8c + 2k, \quad c,k \in \mathbb{R} \end{aligned}$$